MIDTERM #1 (KEY)
JONES, SPRING 1998

Problem 1. *Which of the following formulas is correct...?*

Solution. The correct answer is (c). We have from the product formula

$$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x).$$

Integrating both sides, we get (c).

Problem 2. *If you wanted to expand*

$$\frac{2x+7}{(x+1)^2(x^2+x+19)^2}$$

in partial fractions you would use the sum...:

Solution. The answer is (b). Recall that we put a linear factor in the numerator for each *irreducible quadratic* factor, and a simple constant for the others.

Problem 3. *Which of the following functions* cannot *be integrated in terms of elementary functions...?*

Solution. For (a), we can use integration by parts: in fact, we have

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

For (c), we can use integration by parts twice, then bring the unknown integral to the other side:

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

so that

$$\int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

For (d), we make the substitution $u = \ln x$, so that $du = 1/x \, dx$ and

$$\int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln u + C = \ln(\ln x) + C.$$

For (e), we make the substitution $u = x^2$, so that $du = 2x \, dx$ and

$$\int x \sin(x^2) \, dx = \frac{1}{2}\int \sin u \, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos x^2 + C.$$

The answer is (b).

Problem 4. *Which of the following statements is always correct for a function $f(x)$ with $0 \le f(x) \le C$...?*

Solution. We just test examples to see which statement is true. Take $f(x) = 1/x^2$, for example. Then

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges by the p-series test $(2 > 1)$, but

$$\int_1^\infty \sqrt{\frac{1}{x^2}}\,dx = \int_1^\infty \frac{1}{x}\,dx$$

diverges by the same test. Therefore (a) is out. Similarly,

$$\int_1^\infty (1/x^2)^{-2}\,dx = \int_1^\infty x^4\,dx$$

obviously diverges, so (b) is out.

For (c), do the same analysis above with $f(x) = 1/\sqrt{x}$; this shows that (c) is out. For (e), try the function $f(x) = 1/x$; this shows that (e) is out.

The answer is (d). You can also see this by the comparison test, since

$$f(x) \geq \frac{f(x)}{1+x}$$

for $x \geq 1$, so if $\int_1^\infty f(x)\,dx$ converges, so does the smaller integral $\int_1^\infty f(x)/(1+x)\,dx$.

Problem 5. *Which of the following statements is correct...?*

Solution. The answer is (a). We have

$$|E_S| \leq \frac{K(b-a)^5}{180n^4},$$

so if we double n, the term on the right is changed by a factor of $1/16$.

Statement (b) is false: it is Simpson's rule which is exact on quadratics. Statement (c) is false, certainly there exist functions (like $\sin x^2$) which do not have elementary antiderivatives, so we have to approximate. The error for the midpoint rule is

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $K = \max_{a \leq x \leq b} |f''(x)|$, so (d) is false. Statement (e) is false: in Simpson's rule, we draw quadratic curves fitting the points (it is the trapezoidal rule which uses a linear approximation).

Problem 6. *Find the arc-length function for the curve* $y = x^2/8 - \ln x$ *starting at* $(1, 1/8)$.

Solution. The arc length function is

$$s(x) = \int_1^x \sqrt{1 + f'(t)^2}\,dt$$

where $f(t) = t^2/8 - \ln t$. We have $f'(t) = t/4 - 1/t$, so

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + (t/4 - 1/t)^2}\, dt = \int_1^x \sqrt{1 + t^2/16 - 1/2 + 1/t^2}\, dt \\ &= \int_1^x \sqrt{t^2/16 + 1/2 + 1/t^2}\, dt = \int_1^x \sqrt{(t/4 + 1/t)^2}\, dt \\ &= \int_1^x (t/4 + 1/t)\, dt = t^2/8 + \ln t \Big|_1^x = x^2/8 + \ln x - 1/8. \end{aligned}$$

Problem 7(i). *Evaluate the following indefinite integral:*

$$\int \frac{1}{(x+3)(x-2)}\, dx.$$

SOLUTION. We use partial fractions. We write

$$\frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

and then multiply through to get

$$1 = A(x-2) + B(x+3)$$

With $x = 2$ we see that $5B = 1$, so $B = 1/5$. With $x = -3$ we get $-5A = 1$, so $A = -1/5$. Therefore

$$\begin{aligned} \int \frac{1}{(x+3)(x-2)}\, dx &= -\frac{1}{5} \int \frac{1}{x+3}\, dx + \frac{1}{5} \int \frac{1}{x-2}\, dx \\ &= -\frac{1}{5} \ln|x+3| + \frac{1}{5} \ln|x-2| + C. \end{aligned}$$

Problem 7(ii). *Evaluate the following indefinite integral:*

$$\int (\ln x)^2\, dx.$$

SOLUTION. We use integration by parts, with $u = (\ln x)^2$ so $du = (2 \ln x)/x\, dx$ and $dv = dx$, so $v = x$. We get

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2 \ln x\, dx.$$

Repeating this with now $u = \ln x$ so $du = 1/x\, dx$ and $dv = dx$, $v = x$, we get

$$\int \ln x\, dx = x \ln x - \int dx = x \ln x - x + C.$$

Together this gives

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x \ln x + 2x + C.$$

Problem 7(iii). *Evaluate the following indefinite integral:*

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx.$$

SOLUTION. We substitute $x = 1/2\sin\theta$, so that $\sqrt{1-4x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$, and $dx = 1/2\cos\theta\,d\theta$, so that

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \int \frac{1}{\cos\theta}(1/2)\cos\theta\,d\theta \\ &= \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C \\ &= \frac{1}{2}\sin^{-1} 2x + C.\end{aligned}$$