MIDTERM #1 (KEY)
JONES, FALL 2001

The solutions are included for non-multiple choice problems, except for 3(b).

Problem 1(a)(i). *Evaluate the following integral:*

$$\int_0^3 x^2\sqrt{9-x^2}\,dx.$$

Solution. (Note, some exams had 16 in place of 9 for this problem.)

We substitute $x = 3\sin\theta$, so $\sqrt{9-x^2} = \sqrt{9\cos^2\theta} = 3\cos\theta$, and $dx = 3\cos\theta\,d\theta$. We get

$$\int x^2\sqrt{9-x^2}\,dx = \int (3\sin\theta)^2(3\cos\theta)(3\cos\theta\,d\theta) = 81\int \cos^2\theta\sin^2\theta\,d\theta.$$

Now we note that $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$, so this becomes

$$81\int (1/2\sin(2\theta))^2\,d\theta = \frac{81}{4}\int \sin^2(2\theta)\,d\theta.$$

Now we substitute $\sin^2(2\theta) = (1-\sin(4\theta))/2$ (double the double-angle formula!) to get

$$\frac{81}{4}\int \frac{1-\sin(4\theta)}{2}\,d\theta = \frac{81}{8}\left(\theta+\cos(4\theta)\right)+C.$$

(We could have also replaced $\cos^2\theta = 1-\sin^2\theta$, and then used the double-angle formula twice.)

Now replacing the limits of integration, we have $x=0$ at $\sin\theta = 0$, or $\theta = 0$ and $x = 3$ at $\sin\theta = 1$ or $\theta = \pi/2$, so we have

$$\int_0^3 x^2\sqrt{9-x^2}\,dx = \frac{81}{8}\left(\theta+\cos(4\theta)\right)_0^{\pi/2} = \frac{81}{8}(\pi/2+1-1) = \frac{81\pi}{16}.$$

Problem 1(a)(ii). *Evaluate the integral*

$$\int \frac{x}{x^2-x+6}\,dx.$$

Solution. Note that the denominator does not factor: $(x-3)(x+2) = x^2-x-6 \neq x^2-x+6$. Therefore the method of partial fractions does not apply because the denominator is irreducible qudratic.

Instead, we would like to substitute $u = x^2 - x + 6$, so that $du = (2x-1)\,dx$, hence we let

$$\int \frac{x}{x^2-x+6}\,dx = \frac{1}{2}\int \frac{(2x-1)+1}{x^2-x+6}\,dx$$
$$= \frac{1}{2}\int \frac{2x-1}{x^2-x+6}\,dx + \frac{1}{2}\int \frac{1}{x^2-x+6}\,dx.$$

The first integral is now just

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-x+6| + C.$$

For the second integral, we must complete the square:

$$x^2 - x + 6 = (x-1/2)^2 - 1/4 + 6 = (x-1/2)^2 + 23/4.$$

Hence

$$\frac{1}{x^2-x+6} = \frac{1}{(x-1/2)^2+23/4} = \frac{4/23}{(4/23)(x-1/2)^2+1}.$$

Substituting $u = (2/\sqrt{23})(x-1/2)$, we have $du = 2/\sqrt{23}\,dx$, and

$$\int \frac{1}{x^2-x+6}\,dx = \int \frac{(4/23)(\sqrt{23}/2)}{u^2+1}\,du = \frac{2}{\sqrt{23}}\int \frac{1}{u^2+1}\,du$$
$$= \frac{2}{\sqrt{23}}\tan^{-1} u + C = \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{2}{\sqrt{23}}(x-1/2)\right) + C.$$

Putting these together (remembering the $1/2$), we get

$$\int \frac{x}{x^2-x+6}\,dx = \frac{1}{2}\ln|x^2-x+6| + \sqrt{23}\tan^{-1}\left(\frac{1}{\sqrt{23}}(x-1/2)\right) + C.$$

Problem 1(b). *Evaluate*

$$\int_0^\infty \frac{dx}{x^2-5}$$

or show that it is divergent.

Solution. The function $1/(x^2-5)$ is discontinuous at $x = \sqrt{5}$, therefore we must write

$$\int_0^\infty \frac{dx}{x^2-5} = \lim_{t\to\sqrt{5}^+}\int_0^t \frac{dx}{x^2-5} + \lim_{t\to\sqrt{5}^-}\int_t^\infty \frac{dx}{x^2-5}.$$

Now $x^2 - 5 = (x-\sqrt{5})(x+\sqrt{5})$, so by partial fractions, we have

$$\frac{1}{x^2-5} = \frac{A}{x-\sqrt{5}} + \frac{B}{x+\sqrt{5}}$$

so

$$1 = A(x+\sqrt{5}) + B(x-\sqrt{5}).$$

Letting $x = \sqrt{5}$ we see that $A = 1/(2\sqrt{5})$; letting $x = -\sqrt{5}$ we get $x = -1/(2\sqrt{5})$. Hence

$$\int \frac{dx}{x^2-5} = \int \frac{1/(2\sqrt{5})}{x-\sqrt{5}}\,dx + \int \frac{-1/(2\sqrt{5})}{x+\sqrt{5}}\,dx$$
$$= \frac{1}{2\sqrt{5}}\ln|x-\sqrt{5}| - \frac{1}{2\sqrt{5}}\ln|x+\sqrt{5}| + C = \frac{1}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C.$$

Therefore, for the first integral, say, we have

$$\lim_{t\to\sqrt{5}^+}\int_0^t \frac{dx}{x^2-5} = \lim_{t\to\sqrt{5}^+}\frac{1}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| \to \ln 0 = -\infty$$

so the integral is divergent.

Problem T.*he error in estimating* $\int_a^b f(x)\,dx$ *using Simpson's rule with* n *intervals is at most* $K(b-a)^5 180n^4$ *when* $|f^{(4)}(x)| \le K$ *for* $a \le x \le b$.

How large should n *be to sure the error is less than* 10^{-5} *in estimating*

$$\int_1^3 \frac{\ln x}{2}\,dx$$

using Simpson's rule?

Solution. We have $f(x) = 3/2\ln x$, so $f'(x) = 3/(2x) = 3/2x^{-1}$, $f''(x) = -3/2x^{-2}$, $f'''(x) = 3x^{-3}$, and $f^{(4)}(x) = -9x^{-4} = -9/x^4$. Therefore $|f^{(4)}(x)| = 9/x^4$. This function is clearly decreasing on $1 \le x \le 3$, so it reaches its maximum at $x = 1$, namely, $|f^{(4)}(x)| \le 9$, so we may take $K = 9$.

Therefore

$$|E_S| \le \frac{9(3-1)^2}{180n^4} < 10^{-5}$$

so

$$n^4 > \frac{9(32)(10^5)}{180} = 16(10^4),$$

i.e. $n > \sqrt[4]{16(10^4)} = 20$. Since n must be even in applying Simpson's rule, we take $n = 22$, or any even integer thereafter.

Problem 3(a). *Is the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$$

convergent? If so find its sum.

Solution. The terms of the series $a_n = 1/(n^2+3n+2)$ are positive and decreasing, so the integral test applies. We have
$$\int_1^\infty \frac{1}{x^2+3x+2}\,dx.$$
Since $x^2+3x+2 = (x+2)(x+1)$, we use partial fractions:
$$\frac{1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}$$
and so $1 = A(x+1) + B(x+2)$. Letting $x=-1$ we get $B=1$, $x=-2$ we get $A=-1$, hence
$$\begin{aligned}\int \frac{1}{x^2+3x+2}\,dx &= \int \left(\frac{-1}{x+2} + \frac{1}{x+1}\right)\,dx \\ &= -\ln|x+2| + \ln|x+1| + C = \ln\left|\frac{x+1}{x+2}\right| + C.\end{aligned}$$
Therefore
$$\int_1^\infty \frac{1}{x^2+3x+2}\,dx = \lim_{t\to\infty} \ln\left|\frac{x+1}{x+2}\right|\Bigg|_1^t.$$
We now have:
$$\lim_{t\to\infty} \ln\frac{t+1}{t+2} = \ln\lim_{t\to\infty}\frac{t+1}{t+2} = \ln 1 = 0$$
so the original series is convergent.

From the above we see
$$\sum_{n=1}^\infty \frac{1}{n^2+3n+2} = \sum_{n=1}^\infty \frac{1}{n+1} - \frac{1}{n+2}.$$
This is a telescoping series: we have
$$\sum_{n=1}^N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N+2} = \frac{1}{2} + \frac{1}{N+2}.$$
Therefore
$$\sum_{n=1}^\infty \frac{1}{n^2+3n+2} = \lim_{N\to\infty}\left(\frac{1}{2} + \frac{1}{N+2}\right) = \frac{1}{2}.$$