REVIEW, MIDTERM #1 (Key): MATH 1B

Warning: I have not seen a copy of the midterm examination. Concepts and problems reviewed here should not be taken as an exclusive list! It should give you a good sense, though, of what will be asked.

Problem 1. *The integral*

$$\int_a^\infty \frac{1}{x^{\sqrt{3}}}\,dx$$

converges:

(a) *For all values of* $a > 0$*;*
(b) *For all values of* $a \geq 0$*;*
(c) *For* $a = 1$ *only;*
(d) *For no value of* a*.*

SOLUTION. The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for $p > 1$ and diverges for $p \leq 1$. Since $\sqrt{3} > 1$, the integral converges for any $a > 0$. However, the integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges for $p < 1$ and diverges for $p \geq 1$, so the integral does not converge for $a = 0$—in fact,

$$\int_0^\infty \frac{1}{x^p}\,dx$$

converges for *no* value of p. The answer is (a).

Problem 2. *Evaluate the integral*

$$\int \frac{4x^2+4x+1}{4x^2-4x+1}\,dx.$$

SOLUTION. The degree of the numerator is not less than that of the denominator, so we use long division:

$$\frac{4x^2+4x+1}{4x^2-4x+1} = 1 + \frac{8x}{4x^2-4x+1}.$$

Therefore

$$\int \frac{4x^2+4x+1}{4x^2-4x+1}\,dx = \int dx + \int \frac{8x}{4x^2-4x+1}\,dx = x + \int \frac{8x}{4x^2-4x+1}\,dx.$$

To evaluate the second term, we use partial fractions. First we factor the denominator:

$$4x^2-4x+1 = (2x-1)^2.$$

Therefore we have the partial fractions decomposition

$$\frac{8x}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}.$$

Multiplying through, we obtain

$$8x = A(2x-1) + B = 2Ax + (B-A);$$

we see that $A = 4$ and $B = A = 4$; therefore

$$\int \frac{8x}{(2x-1)^2}\,dx = \int \frac{4}{2x-1}\,dx + \int \frac{4}{(2x-1)^2}\,dx = 2\ln(2x-1) - \frac{2}{2x-1} + C;$$

together this implies that our integral is

$$x + 2\ln(2x-1) - \frac{2}{2x-1} + C.$$

Problem 3. *The area of the surface obtained by rotating the curve $y = f(x)$ from $a \leq x \leq b$ about the x-axis is given by:*

(a) $\int_a^b \pi f(y)^2\,dy$;

(b) $\int_{f(a)}^{f(b)} \pi f(x)^2\,dy$;

(c) $\int_a^b 2\pi y \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dx$;

(d) $\int_{f(a)}^{f(b)} 2\pi x \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dy$.

SOLUTION. Drawing a picture, we see that the radius of the circles we wish to sum up is y, hence

$$A = \int 2\pi y\,ds = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

which is (b).

Problem 4. *Evaluate the integral*

$$\int \sqrt{x} \ln x\,dx.$$

SOLUTION. We use integration by parts, with $u = \ln x$, so $du = 1/x$, and $dv = \sqrt{x}\,dx$, so $v = 2/3x^{3/2}$, and

$$\begin{aligned}\int \sqrt{x} \ln x\,dx &= \int u\,dv = uv - \int v\,du = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}\,dx \\ &= \frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C.\end{aligned}$$

Problem 5. *Which of the following does not have an elementary antiderivative?*

(a) $x^2 \ln x$;
(b) $\sinh x$;
(c) xe^{x^2};
(d) e^{x^2}.

Solution. We can integrate (a) using integration by parts; we in fact obtain

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3,$$

but this is not important. For (b), the antiderivative of $\sinh x$ is $\cosh x$. For (c), we may make the substitution $u = x^2$ to find

$$\int xe^{x^2} \, dx = \frac{1}{2}e^{x^2}.$$

Therefore the answer is (d).

Problem 6. *Evaluate the integral*

$$\int \csc^4 x \, dx.$$

Solution. We first substitute $1 + \cot^2 x = \csc^2 x$, so

$$\int \csc^4 x \, dx = \int (\csc^2 x)(1 + \cot^2 x) \, dx = \int \csc^2 x \, dx + \int \csc^2 x \cot^2 x \, dx.$$

Since $(d/dx)(\cot x) = -\csc^2 x$, the first term is

$$\int \csc^2 x \, dx = -\cot x + C.$$

For the second, make the substitution $u = \cot x$, so then $du = -\csc^2 x \, dx$, hence

$$\int \csc^2 x \cot^2 x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C = -\frac{1}{3}\cot^3 x + C.$$

Putting these together,

$$\int \csc^4 x \, dx = -\cot x - \frac{1}{3}\cot^3 x + C.$$

Problem 7. *If the function $f(x)$ is continuous on $-1 \le x \le 1$ except at the point $x = a$ for some $-1 < a < 1$, then the improper integral*

$$\int_{-1}^{1} f(x)\,dx$$

is written:

(a) $\lim_{t\to a^+} \int_{-1}^{t} f(x)\,dx + \lim_{t\to a^-} \int_{t}^{1} f(x)\,dx$;
(b) $\lim_{x\to a} \int_{-1}^{x} f(x)\,dx$;
(c) $(df/dx)(a) - \lim_{t\to a} \int_{-t}^{t} f(x)\,dx$;
(d) $\lim_{a\to\infty} \int_{-1}^{1} f(x)\,dx$.

Solution. The answer is (a); the rest are nonsense.

Problem 8. *Evaluate the integral*

$$\int \sqrt{x^2-1}\,dx.$$

Solution. This is a problem involving trigonometric substitution: we substitute $x = \sec\theta$, so that $dx = \sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta + 1} = \tan\theta$$

hence

$$\begin{aligned}\int \sqrt{x^2-1}\,dx &= \int \tan\theta(\sec\theta\tan\theta)\,d\theta \\ &= \int \sec\theta\tan^2\theta\,d\theta.\end{aligned}$$

Now integrate by parts, with $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, $dv = \sec\theta\tan\theta\,d\theta$, $v = \sec\theta$, so

$$\begin{aligned}\int \sec\theta\tan^2\theta\,d\theta = \int u\,dv = uv - \int v\,du \\ &= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta)\,d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\tan^2\theta+1)\,d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta - \int \sec\theta\,d\theta.\end{aligned}$$

Since $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$, bringing the integral to the other side, we obtain

$$\sec\theta\tan^2\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|.$$

Replacing this with the values of x, we get

$$\int \sqrt{x^2 - 1}\, dx = \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln \left| x + \sqrt{x^2 - 1} \right| + C.$$

Problem 9. *The sequence*

$$a_n = \frac{2n^2}{n^2+1}$$

is:

(a) *Bounded, monotonic, and convergent;*
(b) *Unbounded, monotonic, and not convergent;*
(c) *Bounded, not monotonic, and convergent;*
(d) *Bounded, not monotonic, and not convergent.*

SOLUTION. The sequence is bounded above by 2 ($n^2+1 > n^2$) and below by 0, say, so it is bounded. It is monotonic increasing, and convergent (it has limit 1, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{1+1/n^2} = 2.$$

Problem 10. *Find the area of the surface obtained by rotating the curve*

$$y = x^2 + 1$$

about the y-axis for $0 \leq x \leq 1$.

SOLUTION. Looking at the graph, we note that the radius of our circles is obtained with radius x, hence we have

$$A = \int 2\pi x\, ds = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx$$

since $dy/dx = 2x$.

We now make a trigonometric substitution, $x = 1/2 \tan\theta$, so that $dx = 1/2 \sec^2\theta\, d\theta$, and

$$\sqrt{1+4x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$$

so that

$$\int x\sqrt{1+4x^2}\, dx = \int \frac{1}{2}\tan\theta \sec\theta \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{4}\int \sec^3\theta \tan\theta\, d\theta.$$

Now let $u = \sec\theta$; then $du = \sec\theta\tan\theta\, d\theta$, so

$$\begin{aligned} \frac{1}{4}\int \sec^2\theta(\sec\theta\tan\theta)\, d\theta &= \frac{1}{4}\int u^2 du = \frac{u^3}{12} + C \\ &= \frac{\sec^3\theta}{12} + C = \frac{(1+4x^2)^{3/2}}{12} + C. \end{aligned}$$

So

$$A = 2\pi \frac{(1+4x^2)^{3/2}}{12}\Bigg|_0^1 = \frac{5\sqrt{5}-1}{6}\pi.$$

Problem 11. *The partial fractions decomposition of*

$$\frac{1}{(x-3)^2(x^2+3)}$$

is:

(a) $\dfrac{A}{x-3}+\dfrac{Bx+C}{(x-3)^2}+\dfrac{Dx+E}{x^2+3}$;
(b) $\dfrac{A}{(x-3)^2}+\dfrac{B}{x^2+3}$;
(c) $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{Cx+D}{x^2+3}$;
(d) $\dfrac{Ax+B}{(x-3)^2}+\dfrac{Cx+D}{x^2+3}$;
(e) $\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{C}{x^2+3}$.

SOLUTION. The answer is (c). Note we only put a linear factor in the numerator when we have an *irreducible quadratic factor* in the denominator.

Problem 12. *Evaluate the integral*

$$\int_0^{\pi} \sin^5 x \, \cos^2 x \, dx.$$

SOLUTION. We let $u = \cos x$, so $du = -\sin x \, dx$, hence

$$\begin{aligned}
\int_0^{\pi} \cos^2 x \sin^4 x \sin x \, dx &= -\int_0^{\pi} u^2(1-u^2)^2 \, du \\
&= -\int_0^{\pi} (u^2 - 2u^4 + u^6) \, du \\
&= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\
&= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.
\end{aligned}$$

Problem 13. *For the integral*

$$\int_0^2 e^x\,dx,$$

given the formula

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

with $|f^{(4)}(x)| \leq K$ *for* $a \leq x \leq b$, *the error in approximating the integral using Simpson's rule for* $n = 2$ *is:*

(a) $|E_S| \leq 1/90$;
(b) $|E_S| \leq e/90$;
(c) $|E_S| \leq e^2/90$;
(d) $|E_S| \leq e^2/2880$.

SOLUTION. We have $f^{(4)}(x) = e^x$, which obtains its maximum on $0 \leq x \leq 2$ at $x = 2$, so $|f^{(4)}(x)| \leq e^2$. Therefore

$$|E_S| \leq \frac{e^2(2^5)}{180(2^4)}$$

which is (c).

Problem 14. *Determine if the integral*

$$\int_1^2 \frac{1}{x\sqrt{\ln x}}\,dx$$

is convergent, and evaluate it if so.

SOLUTION. Make the substitution $u = \ln x$; then $du = 1/x\,dx$, so

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C.$$

The function $1/(x\sqrt{\ln x})$ has a discontinuity at $x = 1$, so we have

$$\int_1^2 \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to 1^-} \int_t^2 \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to 1^-} 2\sqrt{\ln x}\Big|_t^2 = 2\sqrt{\ln 2}.$$

The integral is convergent.

Problem 15. *The sum of the series*

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

is:

(a) $1/2$;
(b) 1;
(c) $2/3$;
(d) $3/2$;
(e) *The series is divergent.*

SOLUTION. This series is

$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1}$$

which is a geometric series: in general,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1$$

so for us, $a = 1$ and $r = -1/2$, so the sum is $1/(1+1/2) = 2/3$. The answer is (c).

Problem 16. *Evaluate*

$$\int_e^{e^2} \frac{\ln(\ln x)}{x}\, dx.$$

SOLUTION. Substitute $t = \ln x$, then $dt = 1/x\, dx$, so

$$\int \frac{\ln(\ln x)}{x}\, dx = \int \ln t\, dt.$$

Now do integration by parts, with $u = \ln t$, $du = 1/t\, dt$, $dv = dt$, $v = t$, so

$$\int \ln t\, dt = \int u\, dv = uv - \int v\, du = t\ln t - \int t(1/t)\, dt = t\ln t - t + C.$$

Hence

$$\int_e^{e^2} \frac{\ln(\ln x)}{x}\, dx = ((\ln x)(\ln(\ln x)) - \ln x)_e^{e^2} = 2\ln 2 - 2 + 1 = 2\ln 2 - 1.$$

Problem 17. *For the integral*

$$\int_a^b (2x^2 + 3x - 4)\, dx,$$

which of the following gives the best approximation for fixed n*?*

(a) *Left endpoint approximation;*
(b) *Right endpoint approximation;*
(c) *Midpoint approximation;*
(d) *The Trapezoidal rule;*
(e) *Simpson's rule.*

SOLUTION. Simpson's rule is *exact* on quadratics (recall, we are fitting a quadratic each time around!). The answer is (e).

Problem 18. *Set up an integral to compute the length of the curve* $y = e^x$ *for* $1 \leq y \leq e$*. Suggest a substitution, but do not evaluate completely.*

SOLUTION. We have

$$L = \int ds = \int_1^e \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Writing $x = \ln y$, we see that $dx/dy = 1/y$, so

$$L = \int_1^e \sqrt{1 + (1/y)^2}\, dy = \int_1^e \frac{\sqrt{y^2+1}}{y}\, dy.$$

We substitute $y = \tan\theta$, so $\sqrt{y^2+1} = \sec\theta$, and $dy = \sec^2\theta$, so

$$\begin{aligned}
\int \frac{\sqrt{y^2+1}}{y}\, dy &= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta \\
&= \int \frac{\sec\theta}{\tan\theta}(1 + \tan^2\theta)\, d\theta \\
&= \int \csc\theta\, d\theta + \int \sec\theta\tan\theta\, d\theta.
\end{aligned}$$

From here, we could finish the evaluation.

Problem 19. *If f is continuous, then*

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx.$$

(a) *True;*
(b) *False;*
(c) *Cannot be determined from the information given.*

SOLUTION. The answer is (b). Recall that $\int_{-\infty}^{\infty} x^3\,dx$, for example, is undefined (divergent), but

$$\lim_{t\to\infty}\int_{-t}^{t} x^3 = \lim_{t\to\infty} 0 = 0.$$

Problem 20. *Determine if the integral*

$$\int_0^1 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$$

is convergent, and evaluate it if so.

SOLUTION. Use the comparison test; since

$$\frac{1}{\sqrt{x}} \le \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}$$

(we just make the numerator a little larger), and

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$$

diverges (by the p-test, with $p = 1/2 \le 1$), the original integral diverges as well.