REVIEW, MIDTERM #2: MATH 1B (Key)

Warning: I have not seen a copy of the midterm examination. Concepts and problems reviewed here should not be taken as an exclusive list! It should give you a good sense, though, of what will be asked.

Problem 1. *The equation*

$$e^x y' = x + y$$

is:

(a) *Linear;*
(b) *Separable;*
(c) *Both linear and separable;*
(d) *Neither linear nor separable.*

SOLUTION. The answer is (a). A separable equation is one of the form

$$\frac{dy}{dx} = f(x)g(y)$$

or equivalently one in which we can put all of the terms involving x on one side and those involving y on the other, something which is not possible in this case.

The equation is linear, however A linear equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

In our case, we have

$$\frac{dy}{dx} - e^{-x}y = xe^{-x}.$$

So the equation is linear.

To the industrious reader, the solution to this equation involves an integral that cannot be solved using elementary functions!

Problem 2. *Evaluate the limit:*

$$\lim_{x\to 0} \frac{(e^{-x} - 1 + x)^3}{x^2(\cos 2x - 1)^2}.$$

SOLUTION. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots.$$

Similarly, we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

so

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} - \dots.$$

Combining these, we have

$$\begin{aligned} \frac{(e^{-x} - 1 + x)^3}{x^2(\cos 2x - 1)^2} &= \frac{(x^2/2 - x^3/6 + \dots)^3}{x^2(-2x^2 + 2x^4/3 - \dots)^2} \\ &= \frac{x^6/8 + \dots}{4x^6 + \dots} = \frac{1/8 + \dots}{4 + \dots} \to \frac{1}{32} \end{aligned}$$

as $x \to 0$.

Problem 3. *Consider the differential equation*

$$y' = -x^2(x^2 + y^2).$$

Which of the following statements is not *true?*

(a) *The solutions to this equation are decreasing functions.*
(b) *Every solution has a critical point.*
(c) *If $y(x)$ is a solution to this equation, then so is $cy(x)$.*
(d) *There are no constant solutions to this equation.*

Solution. The answer is (c). One sees that $y' < 0$ since $x^2, y^2 > 0$, so (a) is true. We also see that (b) is true, since a critical point occurs when $y' = 0 = -x^2(x^2 + y^2)$, for example whenever $x = 0$. One can verify (d) as follows: if $y = c$, then $y' = 0$, and then we'd have $0 = -x^2(x^2 + c^2)$, which holds only for $x = 0$. It is perhaps easier to see why (c) is false: we note that

$$(cy') = cy' = -cx^2(x^2 + y^2) \neq -x^2(x^2 + (cy)^2).$$

Problem 4. *Compute a series expansion for*

$$\int \sqrt{1 + x^3}\, dx.$$

Given an estimate for the integral

$$\int_0^1 \sqrt{1 + x^3}\, dx$$

with an error bounded by $1/160$.

Solution. By the binomial series,

$$\sqrt{1 + x^3} = (1 + x^3)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{3n}$$

with a radius of convergence $|x^3| < 1$, i.e. $|x| < 1$, so

$$\begin{aligned} \int \sqrt{1 + x^3}\, dx &= \int \sum_{n=0}^{\infty} \binom{1/2}{n} x^{3n}\, dx = \sum_{n=0}^{\infty} \binom{1/2}{n} \int x^{3n}\, dx \\ &= \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{3n+1} x^{3n+1} + C. \end{aligned}$$

To simplify this, we note that

$$\begin{aligned} \binom{1/2}{n} &= \frac{(1/2)(1/2 - 1)(1/2 - 2)\dots(1/2 - n + 1)}{n!} \\ &= (-1)^n \frac{1}{2^n} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{n!} \end{aligned}$$

hence

$$\int \sqrt{1+x^3}\,dx = C + x + \frac{1}{8}x^4 + \sum_{n=2}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!(2^n)(3n+1)}x^{3n+1}.$$

Therefore

$$\int_0^1 \sqrt{1+x^3}\,dx = 1 + \frac{1}{8} + \sum_{n=2}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!(2^n)(3n+1)}$$

$$= 1 + \frac{1}{8} - \frac{1}{56} + \frac{1}{160} - \cdots.$$

By the alternating series estimation theorem, then, it is enough to take the terms out to 1/56, i.e.

$$\int_0^1 \sqrt{1+x^3}\,dx \approx 1 + \frac{1}{8} - \frac{1}{56} = \frac{31}{28}.$$

Problem 5. *If* $f(x) = \ln(1 - x^2)$*, what is the value of* $f^{(2n)}(0)$*?*

(a) $1/2n$
(b) $(2n)!/n$
(c) $-1/n$
(d) 0
(e) $-(2n)!/n$
(f) $1/(2n)!$

SOLUTION. We have

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

with radius of convergence $|x| < 1$, so

$$\ln(1 - x^2) = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

with the same radius of convergence.

Recall that this is also the Taylor series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Therefore the coefficient of x^{2n} in the above equation gives the value of $f^{(2n)}(0)/(2n)!$, hence

$$\frac{f^{(2n)}(0)}{(2n)!} = -\frac{1}{n}$$

(don't forget the minus sign!) hence

$$f^{(2n)}(0) = -\frac{(2n)!}{n}.$$

The answer is (e).

Problem 6. *Determine if the series*

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$$

is convergent or divergent.

SOLUTION. The integral test applies; clearly the terms of this sequence are decreasing and positive. We look at

$$\int_2^{\infty} \frac{1}{x \ln x (\ln \ln x)^2} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x (\ln \ln x)^2} \, dx.$$

We first evaluate the indefinite integral

$$\int \frac{1}{x \ln x (\ln \ln x)^2} \, dx.$$

We notice that $u = \ln\ln x$ gives $du = d(\ln x)/\ln x = 1/x\ln x\,dx$, which is exactly what appears! This gives

$$\int \frac{1}{x\ln x(\ln\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C = \frac{-1}{\ln\ln x} + C.$$

Therefore we obtain

$$\lim_{t\to\infty} \frac{-1}{\ln\ln x}\bigg|_2^t = \lim_{t\to\infty} \frac{-1}{\ln\ln t} + \frac{1}{\ln\ln 2} = \frac{1}{\ln\ln 2}$$

since as $t \to \infty$, $\ln t \to \infty$ so $\ln\ln t \to \infty$. By the integral test, this series is convergent.

Problem 7. *The interval of convergence of the series*

$$f(x)=\sum_{n=1}^{\infty}\frac{(x-6)^n}{3^n(n+2)}$$

is:

(a) $|x-6|<3$
(b) $|x|<3$
(c) $[3,9]$
(d) $3\le x<9$
(e) *The series converges for all x.*

Solution. We use the ratio test: we find

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{|x-6|^{n+1}}{3^{n+1}(n+3)}\frac{3^n(n+2)}{|x-6|^n}=\frac{1}{3}|x-6|\frac{n+2}{n+3}\to\frac{1}{3}|x-6|<1$$

as $n\to\infty$. Therefore the series converges for $|x-6|<3$.

We check the endpoints. At $x-6=3$, i.e. $x=9$, we get

$$f(9)=\sum_{n=1}^{\infty}\frac{3^n}{3^n(n+2)}=\sum_{n=1}^{\infty}\frac{1}{n+2}$$

which is divergent as it is a shifted harmonic series (or, to be really on the point, apply the limit comparison test to $\sum_n 1/n$). At $x-6=-3$, i.e. $x=3$, we get

$$f(3)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2}$$

which is a convergent alternating series (the terms $1/(n+2)$ are positive, decreasing, and tend to zero). Therefore the answer is (d).

Problem 8. *Solve the initial value problem*

$$(x^2+1)\frac{dy}{dx}=2xy+4x^2+4,\quad y(0)=0.$$

Solution. Dividing by x^2+1 (which is never zero) and rearranging, we get

$$\frac{dy}{dx}-\frac{2x}{x^2+1}y=\frac{4x^2+4}{x^2+1}=4.$$

We see that

$$\int P(x)\,dx=\int-\frac{2x}{x^2+1}\,dx.$$

We substitute $u=x^2+1$ to get $du=2x\,dx$, hence this becomes

$$-\int\frac{du}{u}=-\ln|u|+C=-\ln|x^2+1|+C=-\ln(x^2+1)+C$$

since $x^2+1>0$. Ignoring C (any one integrating factor will work), we get

$$e^{\int P(x)\,dx} = e^{-\ln(x^2+1)} = e^{\ln(x^2+1)^{-1}} = \frac{1}{x^2+1}.$$

We multiply our first equation (not the original one!) by this factor, to get

$$\frac{d}{dx}\left(\frac{1}{x^2+1}y\right) = \frac{4}{x^2+1}.$$

Hence

$$\frac{y}{x^2+1} = 4\int \frac{1}{x^2+1} = 4\tan^{-1}(x)+C$$

so

$$y(x) = (x^2+1)(4\tan^{-1}(x)+C).$$

We compute

$$y(0) = 1(4\tan^{-1}(0)+C) = C = 0$$

so

$$y(x) = (x^2+1)4\tan^{-1}(x).$$

Problem 9. *Which of the following statements is true?*

(a) *If* $\sum_n c_n 2^n$ *converges, then* $\sum_n c_n(-3)^n$ *converges.*
(b) *If* $\sum_n c_n 2^n$ *converges, then* $\sum_n c_n(-2)^n$ *converges.*
(c) *If* $\sum_n c_n(x-3)^n$ *diverges when* $x = 2$*, then it diverges when* $x = 5$.
(d) *None of the above statements are true.*

SOLUTION. The correct statement is (c). A counterexample for (a) is taking $c_n = (-1/3)^n$, so that

$$\sum_n c_n 2^n = \sum_n (-2/3)^n$$

which is a convergent geometric series, but

$$\sum_n c_n(-3)^n = \sum_n 1^n = \infty.$$

A counterexample for (b) is taking $c_n = (-1/2)^n/n$; one obtains alternatively an alternating harmonic and then harmonic series.

Statement (c) is true, however. There is a radius of convergence for any power series. The series is centered at 3, and it diverges when 2, so its radius of convergence must be smaller than $3 - 2 = 1$. In particular, since $5 - 3 = 2 > 1$, this is also outside its radius of convergence.

Problem 10. *A tank contains* 100 *L of water with a salt concentration of* 0.01 *kg/L. Brine that contains* 0.02 *kg/L of salt enters the tank at* 5 *L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after t minutes?*

SOLUTION. Let $S(t)$ be the amount of salt in the tank in kg after t minutes. Then We have that

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out} = (0.02 \text{ kg/L})(5 \text{ L/min}) - (S(t)/100 \text{ L})(5 \text{ L/min}) \\ &= \frac{1}{10} - \frac{S}{20} = \frac{2-S}{20}.\end{aligned}$$

Separating, we obtain

$$\frac{dS}{2-S} = \frac{1}{20}\,dt$$

so

$$\int \frac{dS}{2-S} = -\ln|2-S| = \int \frac{1}{20}\,dt = \frac{t}{20} + C$$

so that

$$e^{\ln(2-S)} = 2 - S = e^{-t/20+C} = Ce^{-t/20}$$

so finally

$$S(t) = 2 - Ce^{-t/20}.$$

We are told that the tank starts at 0.01 kg/L, so since the tank contains 100 L, the tank starts at 1 kg, i.e.

$$S(0) = 1 = 2 - C$$

so $C = 1$. Putting this together, we get

$$S(t) = 2 - e^{-t/20}.$$

Problem 11. *Calculate the Maclaurin series for*

$$f(x) = \frac{x^2}{(1+x)^2}.$$

Solution. We start with the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Differentiating, we get

$$\frac{-1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n n x^{n-1}.$$

Therefore

$$\frac{x^2}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n+1}.$$

I guess you could also use binomial series; that requires a lot more work!

Problem 12. *Determine if the series*

$$\sum_{n=2}^{\infty}\frac{(-1)^n n}{2n^3 - n - 1}$$

is convergent or divergent.

Solution. This is an alternating series. First we need to verify that

$$b_n = \frac{n}{2n^3 - n - 1}$$

are positive; this is clear because $2n^3 > n + 1$ for all $n \geq 2$. Next we need to show that the b_n are a decreasing sequence; easiest way to do this is to take the derivative: let

$$f(x) = \frac{x}{2x^3 - x - 1}.$$

Then manifestly

$$f'(x) = \frac{(2x^3 - x - 1) - x(6x^2 - 1)}{(2x^3 - x - 1)^2} = \frac{-4x^3 - 1}{(2x^3 - x - 1)^2} < 0.$$

Finally, we check that $b_n \to 0$: we see indeed

$$\lim_{n\to\infty}\frac{n}{2n^3 - n - 1} = \lim_{n\to\infty}\frac{1}{2n^2 - 1 - 1/n} = 0.$$

As it turns out, this series is also absolutely convergent. Taking absolute values,

$$\sum_{n=2}^{\infty}\frac{n}{2n^3 - n - 1}.$$

We can apply the limit comparison test to this series and compare it to $\sum_n (n/n^3) = \sum_n 1/n^2$.

Problem 13. *Which of the following statements is true?*

(a) *If* $\sum_n a_n$ *and* $\sum_n b_n$ *are divergent, then so is* $\sum_n (a_n + b_n)$.
(b) *If* $a_n > 0$ *and* $\lim_{n\to\infty}(a_{n+1}/a_n) < 1$, *then* $\lim_{n\to\infty} a_n = 0$.
(c) *If* $a_n > 0$ *and* $\sum_n a_n$ *is divergent, then* $\sum_n \sqrt{a_n}$ *is convergent.*
(d) *If* $a_n, b_n > 0$, *then* $\sum_n (-1)^n(a_n+b_n) = \sum_n (-1)^n a_n + \sum_n (-1)^n b_n$.
(e) *Mathematics is not a rewarding discipline of study.*

Solution. Statement (a) is false: Take $a_n = n$ and $b_n = -n$; then $\sum_n (a_n+b_n) = \sum_n 0 = 0$. Statement (c) is really false: $\sum_n 1/n$ is divergent, and $\sum_n 1/\sqrt{n}$ is really divergent. Statement (d) is also false, since we may not rearrange the terms of an alternating series! Also, statement (e) is totally fallacious, I don't even know where to start.

Statement (b), however, is true. For if we consider the series $\sum_n a_n$, then $|a_n| = a_n$ since they are positive, and by the ratio test, if the limit $|a_{n+1}/a_n|$ tends to a number less than 1, the series is convergent. If we didn't have $\lim_{n\to\infty} a_n = 0$, then the series would be divergent by the test for divergence!

Problem 14. *Determine if the series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n\sqrt{n^n}}$$

is convergent or divergent.

Solution. We apply the root test: we get

$$|a_n|^{1/n} = \left|\frac{2^n}{n(n^{n/2})}\right|^{1/n} = \frac{2}{n^{1/2}n^{1/n}}.$$

Now $n^{1/n} \to 1$, since

$$\ln \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

(the latter by L'Hopital's rule, for example). Therefore

$$|a_n|^{1/n} \to \frac{2}{n^{1/2}} \to 0$$

so the series is convergent by the root test.

Problem 15. *The series*

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n}$$

is:

(a) *Convergent by the alternating series test.*
(b) *Divergent by the test for divergence.*
(c) *Convergent by the integral test.*
(d) *Convergent by the comparison test.*
(e) *None of the above.*

Solution. This is not an alternating series, since $\sin n$ oscillates between -1 and 1, so (a) is false. The test for divergence fails as well, since clearly $(-1)^n(\sin n)/n \to 0$ as $n \to \infty$ ($|\sin n| \leq 1$). The integral test does not apply, because again, the series is not positive (the terms are also not necessarily decreasing). The comparison test only applies directly to positive series, and if we take absolute values, we obtain $\sum_n |\sin n|/n \leq \sum_n 1/n$ which is divergent, but this tells us nothing. The answer is (e).

Problem 16. *Find the Taylor series for*

$$f(x) = e^{2x}$$

at $x = 1$. What is its radius of convergence? Compute the Taylor polynomial $T_3(x)$ for $f(x)$ at $x = 1$.

Solution. Recall

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that $f(1) = e^2$, $f'(x) = 2e^{2x}$ so $f'(1) = 2e^2$, $f''(x) = 4e^{2x}$ so $f''(1) = 4$, and in general, $f^{(n)}(x) = 2^n e^{2x}$, so $f^{(n)}(1) = 2^n e^2$. Therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n.$$

Alternatively, we can compute this as follows:

$$e^{2(x-1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(2(x-1)\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n.$$

But $e^{2(x-1)} = e^{2x}e^{-2}$, so multiplying through by e^2 gives the above series.

Since the power series for e^x converges everywhere, so does the series for e^{2x} at $x = 1$. Or one can apply the ratio test.

The Taylor polynomial consists of the terms of the series up to degree 3, namely,

$$\begin{aligned} T_5(x) &= e^2 + 2e^2(x-1) + \frac{4e^2}{2}(x-1)^2 + \frac{8e^2}{6}(x-1)^3 \\ &= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3. \end{aligned}$$

Problem 17. *Find the orthogonal trajectory of the family of curves given by*

$$y = \frac{k}{1+x^2}.$$

SOLUTION. We find

$$\frac{dy}{dx} = \frac{-k(2x)}{(1+x^2)^2}.$$

Solving for k we get $k = (1+x^2)y$, so

$$\frac{dy}{dx} = \frac{-2xy}{1+x^2}.$$

Therefore the perpendicular slope is minus the inverse of this, and we want to solve

$$\frac{dy}{dx} = \frac{1+x^2}{2xy}.$$

Separating, we have

$$y\,dy = \frac{1}{2x} + \frac{x}{2}\,dx$$

and integrating we get

$$\frac{y^2}{2} = \frac{1}{2}\ln|x| + \frac{x^2}{4} + C$$

so we get the curves

$$y^2 = \ln|x| + \frac{x^2}{2} + C.$$

Problem 18. *Determine if the series*

$$\sum_{n=1}^{\infty} \ln\left(\frac{1+1/n}{(n+2)/(n+1)}\right)$$

is convergent or divergent. If it is convergent, find its sum.

SOLUTION. This is trickily set up. A bit of staring reveals

$$\begin{aligned}\ln\left(\frac{1+1/n}{(n+2)/(n+1)}\right) &= \ln(1+1/n) - \ln((n+2)/(n+1)) \\ &= \ln(1+1/n) - \ln(1+1/(n+1)).\end{aligned}$$

Therefore the series is telescoping: we have

$$\ln(1+1) - \ln(1+1/2) + \ln(1+1/2) - \ln(1+1/3) + \cdots = \ln 2.$$

The series is convergent. Amazing, huh?