REVIEW, FINAL (Key): MATH 1B
ADDITIONAL PROBLEMS

Warning: I have not seen a copy of the final examination. Concepts and problems reviewed here should not be taken as an exclusive list! It should give you a good sense, though, of what will be asked.

Problem 15. *The integral*

$$\int_a^b f(x)\,dx$$

was approximated using the Trapezoidal rule and $n = 10$. *Using the error bound it was found that* $|E_T| \leq 1$. *Which of the following is the smallest value of* n *in the list for which* $|E_T| \leq 10^{-6}$*?*

(a) 999
(b) 10074
(c) 1053
(d) 60
(e) *The answer cannot be determined from the information given.*

Solution. We know that

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $K = \max_{x\in[a,b]} |f''(x)|$. Therefore the error bound for $n = 10$ gives

$$\frac{K(b-a)^3}{1200} = 1$$

so $K(b-a)^3 = 1200$. Therefore

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{1200}{12n^2} = \frac{100}{n^2} \leq 10^{-6}$$

so $n \geq 10^4 = 10000$. The answer is (b).

Problem 16. *Find the area of the region bounded by the curve*

$$y = \sin^{-1} x$$

and $y = 0$, $x = 1/2$.

Solution. Drawing a picture we see that we are to compute the integral

$$\int_0^{1/2} \sin^{-1} x\,dx,$$

since $y = 0 = \sin^{-1} x$ gives $x = 0$.

To do this integral, we (sneakily) use integration by parts, with $u = \sin^{-1} x$, $dv = dx$, so that $du = 1/\sqrt{1-x^2}\,dx$ and $v = x$, and then we have

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

For the second integral, we substitute $u = 1 - x^2$, so then $du = -2x\,dx$, and hence we have

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \frac{-1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C$$

and in sum

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

Therefore

$$\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}(1/2) + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} - 1 + \frac{1}{2}\sqrt{3}.$$

Problem 17. *Let z_1, z_2 be the solutions to the equation $z^2 - z + 7 = 0$. What is the value of $z_1 + z_2 + 7/(z_1 z_2)$?*

(a) 0
(b) 2
(c) $1 + i$
(d) -1
(e) *None of the above.*

Solution. We have $z = (1 \pm \sqrt{1-28})/2 = 1/2 \pm (3/2)\sqrt{3}i$. Therefore

$$z_1 + z_2 = (1/2 + (3/2)\sqrt{3}i) + (1/2 - (3/2)\sqrt{3}i) = 1$$

and

$$z_1 z_2 = (1/2 + 3/2\sqrt{3}i)(1/2 - 3/2\sqrt{3}i) = 1/4 + 9/4(3) = 7,$$

hence

$$z_1 + z_2 + 7/(z_1 z_2) = 1 + 1 = 2.$$

The answer is (b).

(A previous version had $z^2 - z - 7 = 0$, which had two real roots, something you should already know how to compute with.)

Problem 18. *Solve the differential equation*

$$y'' - 2y' + y = x.$$

Solution. The homogeneous problem $y'' - 2y' + y = 0$ has characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$, so $r = 1$ is a double root, hence $y_h(x) = c_1 e^x + c_2 x e^x$.

We use the method of undetermined coefficients to find the particular solution, and guess $y_p(x) = Ax + B$, so $y_p'(x) = A$ and $y_p''(x) = 0$, so

$$y_p'' - 2y_p' + y_p = 0 - 2A + Ax + B = Ax + (-2A + B) = x$$

So $A = 1$ and $-2A + B = -2 + B = 0$, so $B = 2$. Therefore

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 x e^x + x + 2.$$

Problem 19. *Determine if the series*

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3}$$

is convergent or divergent.

SOLUTION. We use the integral test. The function $f(x) = 1/x(\ln x)^3$ is continuous, positive, and decreasing since $x(\ln x)^3$ is obviously increasing (or just differentiate). Hence we look at

$$\int_3^{\infty} \frac{1}{x(\ln x)^3}\,dx.$$

Substitute $u = \ln x$ to get $du = 1/x\,dx$, then the integral is just

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{du}{u^3} = \frac{-1}{2u^2} = \frac{-1}{2(\ln x)^2}$$

hence

$$\int_3^{\infty} \frac{1}{x(\ln x)^3}\,dx = 0 - \frac{-1}{2(\ln 3)^2} < \infty$$

so the integral and hence the sum are convergent.

Problem 20. *Use Euler's method with step size* $1/2$ *to estimate* $y(1)$ *where* $y(x)$ *is the solution to the intial-value problem* $y' = x + 2y^2$, $y(0) = 0$.

SOLUTION. We start with $x_0 = 0$, $y_0 = 0$, with $h = 1/2$, and

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) = y_{n-1} + \frac{1}{2}(x_{n-1} + 2y_{n-1}^2).$$

Hence

$$y_1 = 0 + \frac{1}{2}(0+0) = 0$$

and

$$y(1) \approx y_2 = 0 + 1/2(1/2+0)^2 = 1/8.$$

Problem 21. *What is the value of*

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n+1}(2n+1)!}?$$

(a) 0
(b) -1
(c) $1/\pi - \pi/2$
(d) $1/\pi + \pi/2$
(e) $(2-\pi)/2\pi$.

SOLUTION. We recognize the series as almost

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

To do this, we first need to add on the $n = 0$ term, which is $(-1)^0 \pi^0 / 2(1)! = 1/2$, hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n+1}(2n+1)!} - \frac{1}{2}.$$

Now the second series is just by multiplying in a π

$$\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{1}{\pi} \sin(\pi/2) = \frac{1}{\pi}$$

so the sum is $1/\pi - 1/2 = (2-\pi)/2\pi$. The answer is (e).

Problem 22. *Evaluate the limit*

$$\lim_{x \to 0} \frac{(\sin 2x - 2x)^2}{x^2(e^x - 1)^3}.$$

SOLUTION. We have $\sin 2x = 2x - (8/3)x^3 + \dots$ and $e^x = 1 + x + \dots$, so we get

$$\frac{(\sin 2x - 2x)^2}{x^2(e^x - 1)^3} = \frac{((-8/3)x^3 + \dots)^2}{x^2(x + \dots)^3} = \frac{(64/9)x^6 + \dots}{x^5 + \dots} = \frac{(16/9)x + \dots}{1 + \dots} \to 0$$

as $x \to 0$.

Problem 23. *Consider the series*

$$\sum_{n=0}^{\infty}(-1)^n \frac{n-1}{n}\sin^2 n.$$

Which of the following statements is true?

(a) *The series is absolutely convergent by the integral test.*
(b) *The series is convergent by the alternating series test.*
(c) *The series is divergent by the test for divergence.*
(d) *The series is convergent by the comparison test.*
(e) *None of the above.*

Solution. The answer is (c), since

$$\lim_{n\to\infty}(-1)^n \frac{n-1}{n}\sin^2 n \neq 0.$$

Statement (a) is false: we end up with $(n-1)/n(\sin^2 n)$ which is positive but not decreasing (it is oscillating, note that $(n-1)/n \to 1$ as $n \to \infty$. Statement (b) is false, again we need the terms $(n-1)/n \sin^2 n$ to be decreasing. Statement (d) is false because the series is divergent.

Problem 24. *Evaluate*

$$\int \frac{dx}{\sqrt{x^2-2x}}.$$

Solution. We need to complete the square in the denominator so we can use a trigonometric substitution. We note that $x^2 - 2x = (x-1)^2 - 1$, so the integral becomes

$$\int \frac{dx}{\sqrt{(x-1)^2-1}} = \int \frac{du}{\sqrt{u^2-1}}$$

where $u = x - 1$. Now we substitute $u = \sec\theta$, so $du = \sec\theta\tan\theta\, d\theta$, and $\sqrt{u^2-1} = \tan\theta$, hence we are left with

$$\begin{aligned}\int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} &= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C \\ &= \ln|u + \sqrt{u^2-1}| + C = \ln|(x-1) + \sqrt{x^2-2x}| + C.\end{aligned}$$

Problem 25. *Consider the sequence defined by*

$$a_n = \frac{ne^{1/n}}{3n-1}.$$

What is

$$\lim_{n\to\infty} a_n?$$

(a) 0
(b) ∞
(c) 1/3
(d) 1
(e) *The sequence does not have a limit.*

SOLUTION. We use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{xe^{1/x}}{3x-1} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^{1/x} + x(-1/x^2)e^{1/x}}{3} = \frac{1+0}{3} = \frac{1}{3}.$$

The answer is (c).

Problem 26. *Evaluate*

$$\int_0^{\pi} \sec x \, dx.$$

SOLUTION. The integral is improper (!) since $\sec \pi/2 = 1/(\cos \pi/2)$ and $\cos \pi/2 = 0$. So we write this as the limit of proper integrals:

$$\int_0^{\pi} \sec x \, dx = \lim_{t\to\pi/2^-} \int_0^t \sec x \, dx + \lim_{t\to\pi/2^+} \int_t^{\pi} \sec x \, dx.$$

We have

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

so

$$\lim_{t\to\pi/2^-} \int_0^t \sec x \, dx = \lim_{t\to\pi/2^-} \ln|\sec t + \tan t| - 0.$$

Now as $t \to \pi/2$, $\sec t \to \infty$ and $\tan t \to \infty$ as well, so $\ln|\sec t + \tan t| \to \infty$. The integral is divergent.

Problem 27. *Find the area of the surface obtained by rotating the parabola* $y = x^2$ *from* $x = 0$ *to* $x = 1$ *around the* y*-axis.*

SOLUTION. We use the formula for surface area

$$A = \int_C r\,ds.$$

Drawing a picture, we see that the radius is x since we are rotating around the y-axis, so we end up with

$$A = \int_0^1 x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 x\sqrt{1+4x^2}\,dx.$$

Substitute $u = 1 + 4x^2$ to get $du = 8x\,dx$, so

$$\int x\sqrt{1+4x^2}\,dx = \int \frac{1}{8}\sqrt{u}\,du = \frac{1}{12}u^{3/2} + C = \frac{1}{12}(1+4x^2)^{3/2} + C.$$

Hence

$$\int_0^1 x\sqrt{1+4x^2}\,dx = \frac{1}{12}(5\sqrt{5} - 1).$$

Problem 28. *Find the particular solution of*

$$y' + y = x + e^x$$

satisfying $y(0) = 0$.

SOLUTION. This is a linear first-order equation. We have $P(x) = 1$ so $I(x) = e^{\int P(x)\,dx} = e^x$, and

$$d(I(x)y) = d(e^x y) = e^x(x + e^x)\,dx$$

so

$$e^x y = \int (xe^x + e^{2x})\,dx = (x-1)e^x + \frac{1}{2}e^{2x} + C,$$

the first integral done by parts. Therefore

$$y(x) = x - 1 + \frac{1}{2}e^x + Ce^{-x}.$$

So

$$y(0) = -1 + 1/2 + C = 0$$

so $C = 1/2$, and

$$y(x) = x - 1 + \frac{1}{2}(e^x - e^{-x}) = x - 1 + \sinh x.$$

Problem 29. *Find the twelfth derivative of* $(x+1)^3e^x$ *at* $x=-1$.

SOLUTION. We must find a Taylor series for e^x at $x=-1$. You can do this by computing derivatives: if $f(x)=e^x$, then $f^{(n)}(x)=e^x$ so $f^{(n)}(-1)=1/e$, hence

$$e^x=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n=\sum_{n=0}^{\infty}\frac{1}{en!}(x+1)^n.$$

Hence

$$(x+1)^3e^x=\sum_{n=0}^{\infty}\frac{1}{en!}(x+1)^{n+3}.$$

Since we want the twelfth derivative, we want the coefficient of $(x+1)^{12}$ in this series, which is $1/(e9!)$. At the same time, if $g(x)=(x+1)^3e^x$ then

$$g(x)=\sum_{n=0}^{\infty}\frac{g^{(n)}(-1)}{n!}(x+1)^n$$

tells us that the coefficient of $(x+1)^{12}$ is $g^{(12)}(-1)/12!$, so equating these two, we get

$$g^{(12)}(-1)=\frac{12!}{e9!}=1320/e.$$

Problem 30. *Determine if the series*

$$\sum_{n=1}^{\infty}\frac{\cos(1/n)}{n}$$

is convergent or divergent.

SOLUTION. Since $\cos(1/n)\to 1$ as $n\to\infty$, we compare the above series to $\sum_{n=1}^{\infty}1/n$. According to the limit comparison test, we should calculate

$$\lim_{n\to\infty}\frac{\cos(1/n)/n}{1/n}=\lim_{n\to\infty}\cos(1/n)=1.$$

Therefore the series diverges since the harmonic series diverges.