REVIEW, FINAL: MATH 1B

Warning: I have not seen a copy of the final examination. Concepts and problems reviewed here should not be taken as an exclusive list! It should give you a good sense, though, of what will be asked.

Problem 1. *Consider the differential equation*

$$y'' + y = \sin^2 x.$$

Which of the following statements is true?

(a) *Every solution $y(x)$ to this equation is concave up.*
(b) $2\cos x + 3\sin x$ *is a solution.*
(c) *A particular solution can be found using $A\sin^2 x$.*
(d) *If $y_1(x)$ and $y_2(x)$ are solutions to the equation, then so is $y_1(x) + y_2(x)$.*
(e) *None of the above.*

Solution. Statement (a) is false, the equation does not tell us anything directly about the second derivative. Statement (b) is false, the linear combination $c_1 \cos x + c_2 \sin x$ is a solution to the homogeneous problem $y'' + y = 0$. Statement (c) is false: the use of undetermined coefficients is generally restricted to polynomials, single powers of sine or cosine, exponentials, and their products. In fact, a long calculation (using variation of parameters) gives

$$y_p(x) = \left(\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\cos x\right)\cos x + \frac{1}{3}\sin^4 x.$$

Actually, this happens to simplify as

$$y_p(x) = \frac{1}{3}(\cos^2 x + 1)$$

if you want to have fun with trigonometric identities. (In a previous version, I didn't check for simplification, so you can actually get a solution if you try something of the form $A\cos^2 x + B\sin^2 x$ and you're down with trig identities.)

Statement (d) is false, as the equation is nonhomogeneous (!): we have

$$(y_1 + y_2)'' + (y_1 + y_2) = (y_1'' + y_1) + (y_2'' + y_2) = 2\sin^2 x.$$

The answer is (e).

Problem 2. *Use series to compute*

$$\int \frac{e^x - 1}{x}\, dx.$$

What is its radius of convergence?

Solution. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots$$

so

$$\frac{e^x - 1}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}.$$

Therefore

$$\int \frac{e^x - 1}{x}\, dx = \int \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \sum_{n=0}^{\infty} \int \frac{x^n}{(n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n+1)!}.$$

The ratio test gives

$$\left| \frac{a_{n+1}}{a_n} \right| = |x| \frac{n}{n+1} \frac{1}{n+1} \to 0$$

as $n \to \infty$, so the radius of convergence is ∞.

Problem 3. *Which of the following functions does* not *have an elementary antiderivative?*

(a) $\dfrac{e^{\tan^{-1} x}}{1+x^2}$
(b) $\sqrt{x^3+1}$
(c) $x^5 e^x$
(d) $\dfrac{\cos(\ln x)}{x}$
(e) $\tan^3 x$.

Solution. The integral (a) can be done by substituting $u = \tan^{-1} x$, we have $du = 1/(1+x^2)\,dx$, and we obtain

$$\int e^u\,du = e^u + C = e^{\tan^{-1} x} + C.$$

The integral (c) can be done through repeated use of integration by parts, a gruesome calculation. For the rock stars among us, the answer is:

$$\int x^5 e^x\,dx = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x.$$

To do the integral in (d), substitute $u = \ln x$ to get $du = 1/x\,dx$ and

$$\int \cos u\,du = \sin u + C = \sin(\ln x) + C.$$

Finally, for (e) we can substitute $\tan^2 x = \sec^2 x - 1$ to obtain

$$\int \tan^3 x\,dx = \int \tan x \sec^2 x\,dx - \int \tan x\,dx.$$

We substitute $u = \tan x$ to get $du = \sec^2 x\,dx$ in the first, and the second integral we know, so we end up with

$$\frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The answer is (b), actually an example of an elliptic integral.

Problem 4. *Consider the differential equation*

$$y'' + xy' - 2y = 0.$$

Suppose that $y(x) = \sum_{n=0}^{\infty} c_n x^n$ *is a solution to this equation satisfying* $y(0) = 0$, $y'(0) = 1$. *Compute the fourth Taylor polynomial for* $y(x)$ *at* $x = 0$.

Solution. We start with

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$
$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$
$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots .$$

Substituting these into the differential equation, we obtain

$$\begin{aligned} y'' + xy' - 2y = (2c_2 + 6c_3 x + 12c_4 x^2 + \dots) + (c_1 x + 2c_2 x^2 + 3c_3 x^3 + \dots) \\ + (-2c_0 - 2c_1 x - 2c_2 x^2 - 2c_3 x^3 - \dots) = 0. \end{aligned}$$

The constant coefficient gives $2c_2 - 2c_0 = 0$, or $c_2 = c_0$. The coefficient on x gives $6c_3 + c_1 - 2c_1 = 0$, so $c_3 = c_1/6$. Finally, the coefficient on x^2 gives $12c_4 + 2c_2 - 2c_2 = 0$, so $c_4 = 0$. We have

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots = c_0 + c_1 x + c_0 x^2 + \frac{c_1}{6} x^3 + 0 + \dots .$$

Now $y(0) = c_0 = 0$, and $y'(0) = c_1 = 1$, so

$$y(x) = x + \frac{1}{6} x^3 + \dots$$

and therefore the fourth Taylor polynomial is

$$T_4(x) = x + \frac{1}{6} x^3.$$

Problem 5. *Consider the antiderivative of a rational function:*

$$F(x) = \int_a^x \frac{P(t)}{Q(t)}\, dt.$$

Which of the following is true?

(a) $F(x)$ *is a rational function.*
(b) $F(x)$ *is a combination of rational functions and logarithms.*
(c) $F(x)$ *is a combination of rational functions, logarithms, and inverse trigonometric functions.*
(d) $F(x)$ *may fail to be an elementary function.*
(e) $\lim_{x\to\infty} F(x) = \infty$.

Solution. The answer is (c). We get rational functions by integrating the likes of $1/t^2$, logarithms from $1/t$, and inverse trig functions from irreducible quadratics $1/(t^2+1)$. It is a theorem that any rational function can be factored in this way, with the denominator consisting of linear and quadratic factors, so in fact we will always obtain an elementary function, hence (d) is false. Answer (e) is nonsense, integrate for example $1/t^2$ to get $F(x) = -1/x + 1/a$, which tends to $1/a$ as $x \to \infty$.

Problem 6. *Use series to estimate the arc length of* $y = (2/5)x^{5/2}$ *for* $0 \le x \le 1$ *with an error* < 0.02.

Solution. We start with the formula for arc length,

$$L = \int_C ds = \int_0^1 \sqrt{1 + (dy/dx)^2}\, dx.$$

We have $dy/dx = x^{3/2}$, so we get

$$L = \int_0^1 \sqrt{1+x^3}\, dx.$$

We must now use binomial series. We have

$$(1+x^3)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{3n}.$$

We compute that

$$\begin{aligned}\binom{1/2}{n} &= \frac{1/2(1/2-1)(1/2-2)\dots(1/2-n+1)}{n!}\\ &= \frac{(-1)^{n-1}(1/2)(1/2)(3/2)\dots((2n-3)/2)}{n!}\\ &= \frac{(-1)^{n-1} 1\cdot 3\cdot\dots\cdot(2n-3)}{2^n n!}.\end{aligned}$$

Therefore

$$\sqrt{1+x^3} = 1 + \frac{1}{2}x^3 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot\dots\cdot(2n-3)}{2^n n!} x^{3n}.$$

Integrating, we obtain

$$\int \sqrt{1+x^3}\,dx = C + x + \frac{1}{8}x^4 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot \cdots \cdot (2n-3)}{2^n n!} \frac{x^{3n+1}}{3n+1}.$$

Hence

$$\begin{aligned} \int_0^1 \sqrt{1+x^3}\,dx &= 1 + \frac{1}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot \cdots \cdot (2n-3)}{2^n n!(3n+1)} \\ &= 1 + \frac{1}{8} - \frac{1}{56} + \cdots . \end{aligned}$$

The series on the end is an alternating series, so we can estimate the sum by taking n terms with error equal to the $(n+1)$th term. Since $1/56 < 0.02$, it suffices to take only the first two terms, and therefore we get the estimate $L = 9/8$.

In fact,

$$\int_0^1 \sqrt{1+x^3} = 1.111447970\ldots$$

so our estimate 1.125 is quite close!

Problem 7. *Let $f(x)$ be a continuous function on the interval $[a, b]$. Which of the following is correct?*

(a) *If $f(x)$ is increasing, then the midpoint approximation gives an upper bound for $\int_a^b f(x)\,dx$.*
(b) *If $f(x)$ is increasing, then the trapezoidal rule gives an upper bound for $\int_a^b f(x)\,dx$.*
(c) *If $f(x)$ is decreasing, then the left endpoint approximation gives an upper bound for $\int_a^b f(x)\,dx$.*
(d) *If $f(x)$ is concave up, then the right endpoint approximation gives a lower bound for $\int_a^b f(x)\,dx$.*
(e) *None of the above.*

Solution. Drawing pictures, we can see that the midterm approximation gives neither an upper nor lower bound, the trapezoidal rule can give both upper and lower bounds depending on the concavity, so (a) and (b) are false. Statement (d) is nonsense, we can have both increasing and decreasing functions which are concave up. The answer is (c).

Problem 8. *Solve the differential equation*

$$y' = xe^x y(y+1).$$

Solution. This equation is separable. We obtain

$$\frac{dy}{y(y+1)} = xe^x\,dx.$$

So we integrate both sides. The left-hand side is the integral

$$\int \frac{dy}{y(y+1)}.$$

We use partial fractions:

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

so $1 = A(y+1) + By$, hence with $y = 0$ we get $A = 1$, $y = -1$ gives $B = -1$, and

$$\int \frac{dy}{y(y+1)} = \int \frac{dy}{y} - \int \frac{dy}{y+1} = \ln|y| - \ln|y+1| = \ln\left|\frac{y}{y+1}\right|.$$

The right-hand side is the integral $\int xe^x\,dx$, which can be done by parts: take $u = x$, $dv = e^x\,dx$, so $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = (x-1)e^x + C.$$

Therefore

$$\ln\left|\frac{y}{y+1}\right| = (x-1)e^x + C$$

so
$$\left|\frac{y}{y+1}\right| = Ce^{(x-1)e^x}$$
and
$$\frac{y}{y+1} = \pm Ce^{(x-1)e^x} = Ce^{(x-1)e^x}.$$
From here, we really should solve for y. Cross-multiplying and simplifying, we get
$$y(1 - Ce^{(x-1)e^x}) = Ce^{(x-1)e^x}$$
so
$$y(x) = \frac{Ce^{(x-1)e^x}}{1 - Ce^{(x-1)e^x}}.$$
This actually simplifies one step further: dividing the top and bottom by the numerator, and replacing $1/C$ by C, we get
$$y(x) = \frac{1}{Ce^{(1-x)e^x} - 1},$$
but don't feel so obligated.

Problem 9. *Which of the following is a solution to $z^4 = -1+\sqrt{3}i$?*

(a) $\sqrt[4]{2}e^{5i\pi/6}$
(b) $\sqrt[4]{2}e^{-2i\pi/3}$
(c) $\sqrt[4]{1/8}(-1+\sqrt{3}i)$.
(d) $\sqrt[4]{2}(-\sqrt{3}+i)$.

SOLUTION. We have the formula that $w^n = z = re^{i\theta}$ has the solutions

$$w_k = r^{1/n}e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1.$$

So for us we have $n = 4$, and $-1+\sqrt{3}i = 2e^{i(2\pi/3)}$, so

$$w_k = \sqrt[4]{2}e^{i(2\pi/3+2\pi k)/4} = \sqrt[4]{2}e^{i(\pi/6+\pi k/2)}.$$

Hence

$$w_0 = \sqrt[4]{2}e^{i\pi/6} = \sqrt[4]{2}(\cos\pi/6 + i\sin\pi/6) = \sqrt[4]{2}(\sqrt{3}/2 + i/2)$$

and

$$w_1 = \sqrt[4]{2}e^{4i\pi/6} = \sqrt[4]{2}(-1/2+\sqrt{3}i/2)$$

which is answer (c).

Problem 10. *Evaluate*

$$\int_{-1}^{1} \frac{1}{\sqrt{4x^2-x^4}}\,dx.$$

SOLUTION. Notice that this integral is improper! At $x = 0$, the function $1/\sqrt{x^4-4x^2} = 1/x\sqrt{4-x^2}$ blows up. In particular

$$\int_{-1}^{1} \frac{1}{x\sqrt{4-x^2}}\,dx = \lim_{t\to 0^-}\int_{-1}^{t} \frac{1}{x\sqrt{4-x^2}}\,dx + \lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x\sqrt{4-x^2}}\,dx.$$

To test for convergence, let us evaluate the indefinite integral. We use the trigonometric substitution $x = 2\sin\theta$, so $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$, and $dx = 2\cos\theta\,d\theta$. We get

$$\int \frac{1}{x\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{2\sin\theta(2\cos\theta)}\,d\theta = \frac{1}{2}\int \csc\theta = \frac{1}{2}\ln|\csc\theta - \cot\theta| + C.$$

We find $\csc\theta = 1/\sin\theta = 2/x$, and $\cot\theta = \cos\theta/\sin\theta = \sqrt{4-x^2}/x$, so this becomes

$$\int \frac{1}{x\sqrt{4-x^2}}\,dx = \frac{1}{2}\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + C.$$

And

$$\lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x\sqrt{x^2-4}}\,dx = \frac{1}{2}\ln(2-\sqrt{3}) - \lim_{t\to 0^+}\frac{1}{2}\ln\left|\frac{2-\sqrt{4-t^2}}{t}\right|.$$

As a last step, we have to apply L'Hopital's rule: we get

$$\lim_{t\to 0}\frac{2-\sqrt{4-t^2}}{t} = \lim_{t\to 0}\frac{t}{\sqrt{4-t^2}} = 0,$$

so the above limit tends to $-\ln 0 = -(-\infty) = \infty$, so the integral is divergent.

Problem 11. *For which value of m do we have*

$$\int_0^\pi \cos^2 mx \sin mx\, dx = 2/3?$$

(a) 0
(b) 2
(c) -1
(d) 1
(e) *None of the above.*

SOLUTION. Clearly $m \neq 0$. We substitute $u = \cos mx$, so that $du = -m \sin mx\, dx$, and then the integral is

$$\int \cos^2 mx \sin mx\, dx = -\frac{1}{m}\int u^2\, du = \frac{-1}{3m}\cos^3 mx + C$$

and

$$\int_0^\pi \cos^2 mx \sin mx\, dx = \frac{-1}{3m}\cos^3 mx\Big|_0^\pi = \frac{-1}{3m}(\cos^3 m\pi - 1) = 1.$$

The choice $m = 1$ gives $-1/3(-1-1) = 2/3$, so the answer is (d).

(In a previous version, I had $1/2$ as a possibility, which also happens to be a solution.)

Problem 12. *The punch at a party is kept in a* 10 L *bowl and is initially half-full of cranberry juice. Some unruly partygoers decide to spike the punch with PCP and are able to slip a conconction with a toxicity of* 5 mg/L *into the punch at a rate of* 1 L/min. *The punch is being drunk at a rate of* 1/2 L/min. *When the bowl is full, how toxic is the punch?*

SOLUTION. Let $p(t)$ be the amount of PCP in milligrams in the punch bowl after t minutes. Then since there is $5 + t/2$ liters of liquid in the bowl at time t, we have

$$\frac{dp}{dt} = 5 - \frac{p}{5 + t/2}(1/2) = 5 - \frac{p}{10+t}.$$

Rewriting, this is a linear first-order differential equation:

$$\frac{dp}{dt} + \frac{1}{10+t}p = 5.$$

The integrating factor is $e^{\int dt/(10+t)} = e^{\ln(10+t)} = 10 + t$, so we obtain

$$d((10+t)p) = 5(10+t)\, dt.$$

Integrating both sides, we obtain

$$p(t) = \frac{(5/2)t^2 + 50t + C}{t+10}.$$

Since $p(0) = C/10 = 0$, we see $C = 0$, so

$$p(t) = \frac{5t^2 + 100t}{2t + 20}.$$

The bowl fills at $t = 10$, which gives $p(10) = \frac{1500}{40} = 37.5$ mg.

Problem 13. *Which of the following statements is* in*correct?*

(a) *If $a_n, b_n > 0$, $\sum_n a_n$ converges and the sequence $\{b_n/a_n\}$ converges, then $\sum_n b_n$ converges.*
(b) *If $a_n > 0$ for all n and $a_n \to 0$, then $\sum_n (-1)^n a_n$ converges.*
(c) *If $a_n > 0$ for all n and $\sum_n a_n$ converges, then $\sum_n (-1)^n a_n$ converges.*
(d) *If $\sum_n a_n$ converges then $\sum_n a_n/2^n$ converges.*
(e) *If $a_n > 0$ and $\sum_n a_n$ converges then $\sum_n (-1)^n a_n^2$ converges absolutely.*

Solution. Statement (a) is the extended version of the limit comparison theorem, so it is true. Statement (c) is definitely true, the alternating series is, in fact, absolutely convergent, hence the series itself is convergent. For (d), we consider the power series $\sum_n a_n x^n$. Since it converges when $x = 1$, its radius of convergence is $R \geq 1$. Therefore the series converges at $x = 1/2$, hence (d) is true. Statement (e) is true: if $\sum_n a_n$ converges, then $a_n \to 0$, so at some point $a_n < 1$, so $a_n > a_n^2$, hence the series $\sum_n a_n^2$ is convergent by comparison to $\sum_n a_n$. (In a previous version, I left out the condition that $a_n > 0$, which makes the above inequality false!)

Statement (b) is in fact false. Consider the series

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \dots.$$

The partial sums of this series tend to zero, as the sum of the negative terms is convergent (a geometric series), whereas the sum of the positive terms is not (it is the harmonic series).

Problem 14. *Determine if the series*

$$\sum_{n=1}^{\infty} (-1)^n (\sqrt[n]{n} - 1)^n$$

is convergent or divergent.

Solution. We use the root test: we get

$$\sqrt[n]{|a_n|} = \sqrt[n]{n} - 1.$$

Now recall that $n^{1/n} \to 1$ as $n \to \infty$, as

$$\ln n^{1/n} = \frac{\ln n}{n} \to 1/n \to 0$$

by L'Hopital's rule, so $n^{1/n} \to e^0 = 1$. Therefore

$$\sqrt[n]{|a_n|} \to 1 - 1 = 0,$$

so the series is convergent.