MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #1

Problem 1 (DF 0.1.5). Determine whether the following functions f are well-defined:

(a) $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f(a/b) = a$;
(b) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(a/b) = a^2/b^2$.

Problem 2. For $a, b \in \mathbb{R}$, define $f_{a,b} : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + ax + b$. Prove that for every a, b, the map $f_{a,b}$ is neither injective nor surjective.

Problem 3. Let $f : A \to B$ and $g : B \to C$ be maps. Suppose that $g \circ f : A \to C$ is injective. Show that f is injective. Is g necessarily injective? Give a proof or a counterexample.

Problem 4. Let X be a set. For subsets $A, B, C \subset X$ we define

$$A \oplus B = (A \setminus B) \cup (B \setminus A) \quad \text{and} \quad A \odot B = A \cap B.$$

Prove that:

(a) $A \oplus B = B \oplus A$;
(b) $A \oplus \emptyset = A$;
(c) $A \odot A = A$;
(d) $A \odot (B \oplus C) = (A \odot B) \oplus (A \odot C)$.

Problem 5 (DF 0.1.7). Let $f : A \to B$ be a surjective map of sets. Prove the relation

$$a \sim b \quad \Longleftrightarrow \quad f(a) = f(b)$$

is an equivalence relation whose equivalence classes are the fibers of f.

Problem 6 (DF 0.2.1(a)–(c)). For each of the following pairs of integers a and b, determine their greatest common divisor $\gcd(a, b)$, their least common multiple $\operatorname{lcm}(a, b)$, and write their greatest common divisor in the form $ax + by$ for some integers x and y.

(a) $a = 20$, $b = 13$.
(b) $a = 69$, $b = 372$.
(c) $a = 792$, $b = 275$.

Problem 7 (DF 0.2.7). If p is a prime, prove that there do not exist nonzero integers a and b such that $a^2 = pb^2$ (i.e., $\sqrt{p}$ is not a rational number).

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