MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #2

Problem 1 (DF 0.3.4). Compute the remainder when 37^{100} is divided by 29.

Problem 2 (DF 0.3.13–14). Let $n \in \mathbb{Z}_{>1}$ and let $a \in \mathbb{Z}$. Prove that $\gcd(a,n) = 1$ if and only if there exists $c \in \mathbb{Z}$ such that $ac \equiv 1 \pmod{n}$.

Problem 3. Let X be a set and let $*$ be a binary operation on X. Suppose that:

(i) $a * b = b * a$ for all $a, b \in X$, and
(ii) $a * b = a$ for all $a, b \in X$.

Show that X can have at most one element.

Problem 4. Determine which of the following are groups. Justify your answer.

(a) The set $G = \mathbb{R} \setminus \{0\}$ under the binary operation $*$ defined by $a * b = a/b$ for $a, b \in G$;
(b) The set $G = \mathbb{R}$ under the binary operation $*$ defined by $a * b = a + b + ab$ for $a, b \in G$;
(c) The set $G = \{z \in \mathbb{C} : z^n = 1 \text{ for some } n \in \mathbb{Z}_{>0}\}$ under multiplication.

Problem 5 (DF 1.1.12). Find the orders of the following elements of the multiplicative group $(\mathbb{Z}/12\mathbb{Z})^\times$: $\overline{1}$, $\overline{-1}$, $\overline{5}$, $\overline{7}$, $\overline{-7}$, $\overline{13}$.

Problem 6 (DF 1.1.20). Let G be a group and let $x \in G$. Show that x and x^{-1} have the same order.

Problem 7. Prove that if G is a finite group with even order, then there must be an element $a \neq e$ such that $a = a^{-1}$.

Date: 5 September 2007; due Wednesday, 12 September 2007.