MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #4

Problems (for all)

Problem 1 (DF 1.4.10). Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}, ac \neq 0 \right\}.$$

Show that G is a group under matrix multiplication.

Problem 2. In this exercise, we show that each of the following groups are mutually non-isomorphic:

$$\mathbb{Z}/8\mathbb{Z}, \quad (\mathbb{Z}/16\mathbb{Z})^\times, \quad D_8, \quad Q_8.$$

(a) Show that each of these groups has order 8.
(b) Show that if G, H are groups and $G \cong H$, then G is abelian if and only if H is abelian. Conclude that neither of the first two groups is isomorphic to either of the last two groups.
(c) Show that $\mathbb{Z}/8\mathbb{Z}$ has an element of order 8 but $(\mathbb{Z}/16\mathbb{Z})^\times$ has no element of order 8. Conclude that $\mathbb{Z}/8\mathbb{Z} \not\cong (\mathbb{Z}/16\mathbb{Z})^\times$.
(d) Show that every element in $Q_8 \setminus \{1, -1\}$ has order 4. Conclude that $D_8 \not\cong Q_8$.

Problem 3. Let $m, n \in \mathbb{Z}$ with $n > 1$. Consider the map:

$$\begin{aligned} \phi : \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/n\mathbb{Z} \\ a &\mapsto ma. \end{aligned}$$

(a) Show that ϕ is a homomorphism.
(b) Show that if $\gcd(m, n) = 1$, then m is an isomorphism.

Problem 4. Let G be a group. Show that the map $\phi : G \to G$ defined by $\phi(a) = a^2$ is a homomorphism if and only if G is abelian.

Extra problems (for grad students)

Problem 5 (DF 1.4.7). Let p be prime. Show that $\#GL_2(\mathbb{F}_p) = p(p-1)(p^2-1)$. [Hint: Subtract the number of 2×2-matrices which are *not* invertible from the total number of 2×2 matrices over $\mathbb{F}_p$. You may use the fact that a 2×2-matrix is not invertible if and only if one row (or column) is a multiple of the other.]

Date: 19 September 2007; due Wednesday, 26 September 2007.