MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #5

Problems (for all)

Problem 1 (DF 1.7.21). Show that the group of rigid motions of a cube is isomorphic to S_4. [Hint: Show that the group of rigid motions acts on the set of four pairs of opposite vertices.]

Problem 2.

(a) Let F be a field and $n \in \mathbb{Z}_{>0}$. Define the *special linear group*

$$SL_n(F) = \{A \in GL_n(F) : \det(A) = 1\}.$$

Show that $SL_n(F)$ is a subgroup of $GL_n(F)$.

(b) Show that the set

$$H = \{a + b\sqrt{2} : a, b \in \mathbb{Q},\ a, b \text{ not both zero}\}$$

is a subgroup of $\mathbb{R}^\times$ under multiplication.

(c) Let $H = \{x \in \mathbb{R} : x^2 \in \mathbb{Q}\}$. Show that H is not a subgroup of $\mathbb{R}$ under addition.

Problem 3. Let H, K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

Problem 4 (sorta DF 2.1.6).

(a) Let G be an abelian group, and let

$$G_{\text{tors}} = \{g \in G : g \text{ has finite order}\}.$$

Show that G_{tors} is a subgroup of G, known as the *torsion subgroup*.

(b) Show that if G is a finite abelian group, then $G_{\text{tors}} = G$.

(c) Determine the following groups: $\mathbb{Z}_{\text{tors}}$, $(\mathbb{Q}^\times)_{\text{tors}}$, and $(\mathbb{C}^\times)_{\text{tors}}$.

Problem 5 (DF 2.2.4). For $G = Q_8$, compute the center $Z(G)$ and for each $a \in G$, its centralizer $C_G(\{a\})$.

Extra problems (for grad students)

Problem 6. Let G be a group. Show that the map

$$G \times G \to G$$
$$(g, a) \mapsto g \cdot a = gag^{-1}$$

defines a group action of G on itself.

Problem 7 (DF 2.1.13). Let H be a subgroup of $\mathbb{Q}$ under addition with the property that $1/x \in H$ for all nonzero $x \in H$. Show that $H = 0$ or $H = \mathbb{Q}$.

Date: 26 September 2007; due Wednesday, 3 October 2007.