MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #8

Problems (for all)

Problem 1. In this problem, we prove the "interesting" parts of the fourth isomorphism theorem. Throughout, let G be a group, let $N \trianglelefteq G$ be a normal subgroup, and let $H \leq G$ be a subgroup such that $H \supset N$.

(a) Define $H/N = \{hN : h \in H\}$. Show that $H/N \leq G/N$ is a subgroup.
(b) Show that if $H \trianglelefteq G$ is normal, then $H/N \trianglelefteq G/N$ is normal.

Problem 2 (DF 3.5.3). Prove that S_n is generated by the set $\{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$. [Hint: Consider conjugates, e.g. $(2\ 3)(1\ 2)(2\ 3)^{-1}$.]

Problem 3 (DF 3.5.8). Write out the subgroup lattice for A_4 (see DF Figure 8, page 111—but only after you try yourself!), and justify your work. For each subgroup N which is normal, determine the isomorphism class of N and A_4/N.

Problem 4 (DF 4.2.3(a)). Let $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ and label these with the integers $1, 2, \ldots, 8$. Exhibit the image of each element of D_8 under the left regular representation of D_8 into S_8.

Problem 5 (DF 4.3.11(a)(c)(d)). For each $\sigma_1, \sigma_2 \in S_n$ below, determine if σ_1, σ_2 are conjugate. If they are, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

(a) $\sigma_1 = (1\ 2)(3\ 4\ 5)$ and $\sigma_2 = (1\ 2\ 3)(4\ 5)$.
(b) $\sigma_1 = (1\ 5)(2\ 3\ 7)(6\ 8\ 11\ 10)$ and $\sigma_2 = \sigma_1^3$.
(c) $\sigma_1 = (1\ 3)(2\ 4\ 6)$ and $\sigma_2 = (3\ 5)(2\ 4)(5\ 6)$.

Problem 6 (sorta DF 4.3.2). Find all conjugacy classes in the groups D_8, Q_8, $\mathbb{Z}/8\mathbb{Z}$.

Problems (for grad students)

Problem 7. Show that every element in A_n for $n \geq 3$ can be written as the product of (not necessarily disjoint) 3-cycles.

Problem 8 (DF 4.3.25). Let $G = GL_2(\mathbb{C})$, and let $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}, ac \neq 0 \right\}$. Prove that every element of G is conjugate to some element of the subgroup H and deduce that G is the union of conjugates of H. *[Hint: Show that every element of $GL_2(\mathbb{C})$ has an eigenvector.]*

Only 11 homeworks total, so 3 more to go!

Date: 24 October 2007; due Wednesday, 31 October 2007.