MATH 251: ABSTRACT ALGEBRA I
HOMEWORK #11

Problems (for all)

Problem 1. Let I, J be ideals of a ring R.

(a) Show that $I \cap J$ is an ideal of R.

(b) Define the *sum* of I and J to be
$$I + J = \{a + b : a \in I, b \in J\}.$$
Prove that $I + J$ is the smallest ideal of R containing both I and J. *[Hint: Show that $I + J$ is an ideal, that $I, J \subset I + J$, and that if N is an ideal containing both I and J then $I + J \subset N$.]*

Problem 2. Let R be a commutative ring and let
$$M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \right\}$$
be the 2×2-matrix ring over R.

(a) Let $I \subset R$ be an ideal. Show that
$$M_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in I \right\}$$
is an ideal of $M_2(R)$.

(b) Let
$$J = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} : x, y \in R \right\}.$$
Show that J is a left ideal of $M_2(R)$ but not a right ideal.

Problem 3. Let $x^2 + x + 1 \in \mathbb{F}_2[x]$ and let $R = \mathbb{F}_2[x]/I$ be the quotient ring by the ideal $I = (x^2 + x + 1)$. Denote the quotient map
$$\mathbb{F}_2[x] \to R = \mathbb{F}_2[x]/I$$
$$x \mapsto x + I = \overline{x}.$$

(a) Show that $R = \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\}$ has four elements.

(b) Show that the additive group of R is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) Show that the multiplicative group $R^\times$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Deduce that R is a field.

Problem 4 (DF 7.4.4). Let R be a commutative ring. Prove that R is a field if and only if $(0) \subset R$ is a maximal ideal.

Problem 5 (DF 7.5.4). Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

Date: 28 November 2007; due Wednesday, 5 December 2007.

Problems (for grad students)

Problem 6 (DF 7.3.2). Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Problem 7 (sorta DF 7.3.34). Let I, J be ideals of a ring R.

(a) Define the *product* of I and J to be

$$IJ = \left\{\sum_{i=1}^n a_i b_i : a_i \in I, b_i \in J, n \in \mathbb{Z}_{\geq 0}\right\}.$$

Show that IJ is an ideal of R.

(b) Show that $I \cap J \supset IJ$.

(c) Give an example where $IJ \neq I \cap J$.