MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
EXAM #2

Problem 1. Statement (a) is true: if A is countable then A is nonempty, and if $a \in A$ then there exists a neighborhood U of a with $U \subset A$; but any open interval is uncountable, so A is uncountable. Statement (b) is false: the sets $[0, 1]$ and $[1, 2]$ are perfect (closed and have no isolated points) but $[0, 1] \cap [1, 2] = \{1\}$ consists of a single isolated point, so is not perfect. Statement (c) is false: for example, $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous (and $\mathbb{R}$ is closed!). Statement (d) is false: for example, $f(x) = x^2$ on $[-1, 1]$ has $f([-1, 1]) = [0, 1] \not\subset [1, 1] = \{1\}$. Finally, (e) is false (we need f to be continuous on $[a, b]$): for example, take $f(x) = x$ on $(0, 1)$ but $f(0) = f(1) = 0$.

Problem 2. Let x be a limit point of S. Then there exists a sequence $x_n \to x$ with $x_n \in S$. Since A is closed and $S \subset A$ we conclude $x \in A$. Since f is continuous on A, we have $f(x_n) \to f(x)$. Since $f(x_n) \geq 5$, by the order limit theorem, we have $f(x) \geq 5$ as well, so $x \in S$ and hence S is closed.

Problem 3. We claim

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Let $\epsilon > 0$. Let $\delta = \epsilon$. If $x < 0$ then $f(x)/x = 0$. If $0 < x < \delta = \epsilon$ then $f(x)/x = x^2/x = x < \epsilon$. In either case, we have that $0 < |x| < \delta$ implies $|f(x)/x - 0| = |f(x)/x| < \epsilon$, which proves (a).

For (b), by calculus and the above we have

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Therefore $f'(x)$ is continuous whenever $x \neq 0$; when $x = 0$ we claim it is also continuous, indeed given $\epsilon > 0$ let $\delta = \epsilon/2$ then $|x| < \delta = \epsilon/2$ implies $|f(x)| \leq \max(0, 2|x|) = 2|x| < \epsilon$. However, f' is not differentiable at $x = 0$, since

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{x};$$

if $x_n \to 0$ with $x_n > 0$ then $\lim f'(x_n)/x_n = \lim 2x_n/x_n = 2$; if $x_n \to 0$ with $x_n < 0$ then $\lim f'(x_n)/x_n = 0 \neq 2$. So by the sequential criterion, the limit $\lim_{x\to 0} f'(x)/x$ does not exist so f' is not differentiable.

Problem 4. Let $\epsilon > 0$. Let $\delta = \sqrt{\epsilon/\lambda}$. Then if $|x-y| < \delta = \sqrt{\epsilon/\lambda}$ and $x, y \in A$ then $|x-y|^2 < \epsilon/\lambda$ so

$$|f(x) - f(y)| \leq \lambda |x - y|^2 < \lambda \frac{\epsilon}{\lambda} = \epsilon.$$

So f is uniformly continuous on A.

Problem 5. For (a), suppose $K_1, \dots, K_n$ are compact. Then each K_i is closed and bounded. The finite union of closed sets is closed so $K = K_1 \cup \cdots \cup K_n$ is closed. If $|x_i| \leq M_i$ for all $x_i \in K_i$ then $|x| \leq M = \max(\{M_1, \dots, M_n\})$ for all $x \in K$, so K is bounded. Thus K is compact.

For (b), let $K_n = \{n\}$. Then $\bigcup_{n=1}^{\infty} K_n = \mathbb{N}$ which is closed but not bounded, and in any case not compact.

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