MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
FINAL EXAM

Name ________________________________

Problem 1. Mark each as true or false. Briefly justify your answer.

(a) If $x_n \to x$ and $f : \mathbb{R} \to \mathbb{R}$ is a bounded function then $f(x_n)$ converges.

(b) There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ -1, & \text{if } x < 0. \end{cases}$$

(c) If $A \subset \mathbb{R}$ is open and $A \supset \mathbb{Q}$, then $A = \mathbb{R}$.

(d) If $K \subset \mathbb{R}$ is compact, then K is connected.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded then f attains a minimum and maximum value.

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Problem 2. Show using the definition that the sequence (a_n) with $a_n = \dfrac{n}{2n+1}$ converges.

Problem 3. Let $A, B \subset (0, \infty)$ be subsets which are bounded above. Let

$$AB = \{ab : a \in A, b \in B\}.$$

Show that

$$\sup AB = (\sup A)(\sup B).$$

Problem 4. Define a sequence (a_n) by $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{5}{a_n}$. Prove that the sequence converges and find its limit.

Problem 5. Mark each as true or false. Briefly justify your answer.

(a) There exists a subset of $\mathbb{R}$ with exactly four limit points.

(b) Every subset of $\mathbb{R}$ is either countable or uncountable.

(c) If (f_n) and (g_n) are uniformly convergent, then $(f_n g_n)$ is uniformly convergent.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is compact, then $f^{-1}(A)$ is closed.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is compact, then $f(A)$ is closed.

Problem 6. Suppose that $\sum_{n=1}^{\infty} a_n$ converges with $a_n \geq 0$. Prove that for all $\epsilon > 0$, there exists a subsequence (b_n) of (a_n) such that $\sum_{n=1}^{\infty} b_n < \epsilon$.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 + x, & \text{if } x \geq 0; \\ \dfrac{1}{1-x}, & \text{if } x < 0. \end{cases}$$

(a) Prove that f is differentiable.

(b) Is f' continuous? (A brief justification will suffice.)

Problem 8. Show that the sequence of functions

$$f_n(x) = \frac{x^n}{1 + x^n}$$

converges pointwise on $[0, 1]$. Does it converge uniformly on $[0, 1]$?

Problem 9. Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n^2$ converges (absolutely).

Problem 10. Let $f : [-1, 2] \to \mathbb{R}$ be a continuous function. Suppose that $f(-1) = 0$ and $f(2) = 5$. Show that $f(x) = x^2$ for some $x \in (-1, 2)$.

Problem 11. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f'(x)$ is bounded on (a, b). Use the mean value theorem to prove that f is uniformly continuous.

Problem 12. Let $f : [a, b] \to \mathbb{R}$ be one-to-one and continuous, and suppose that $f(a) < f(b)$. Show that $f([a, b]) = [f(a), f(b)]$.