MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
HOMEWORK #8

Problems (for all)

4.2.1(a)(c)
4.2.A: Let $t : \mathbb{R} \to \mathbb{R}$ denote the Thomae function. Show that $\lim_{x \to 1} t(x) = 0$. *[Hint: See Exercise 4.2.4.]*
4.2.6
4.2.9
4.3.2
4.3.4
4.3.7
4.3.12

Problems (for grad students)

4.3.A: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = kx$ for some $k \in \mathbb{R}$. Conclude that the only continuous group automorphism f of $\mathbb{R}$ with $f(1) = 1$ (hence the only continuous ring automorphism of $\mathbb{R}$) is the identity. *[Hint: See Exercise 4.3.10.]*
4.3.B: Let $f : [a, b] \to \mathbb{R}$ be an *increasing* function, i.e. $f(x) \le f(y)$ whenever $x \le y$.

(a) Let $c \in [a, b]$. Show that $\lim_{x \to c^-} f(x)$ exists, i.e. for all sequences (x_n) with $x_n \to x$ and $x_n < x$ for all n, the limit $\lim f(x_n)$ exists and this limit is independent of the choice of sequence (x_n). Conclude similarly that $\lim_{x \to c^+} f(x)$ exists.
(b) Define the *jump* function $j : [a, b] \to \mathbb{R}$ defined by $j(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$. Show that $j(c) \ge 0$ for all $c \in [a, b]$. For any $\epsilon > 0$, show that there are at most finitely many points $c \in [a, b]$ such that $j(c) \ge \epsilon$.
(c) Conclude that if f is a monotone function on $[a, b]$ then f has at most countably many discontinuities, i.e. the set $\{c \in [a, b] : f \text{ is not continuous at } c\}$ is finite or countable.

Date: due Wednesday, 4 November 2009.