MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
FINAL EXAM

Name ______________________________

Problem	**Score**	**Problem**	**Score**
1		6	
2		7	
3		8	
4		9	
5			

Total __________

Date: Friday, 17 December 2010.

Problem 1. Let $t \in \mathbb{R}$. Compute
$$\lim_{n\to\infty} \frac{1}{tn+1}$$
and prove that your answer is correct using the definition.

Problem 2. Mark each as true or false. Briefly justify your answer.

(a) The map

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$$
$$d(x, y) = |x - y|^2$$

is a metric on $\mathbb{R}$.

(b) For $n \in \mathbb{N}$, let $A_n \subseteq \mathbb{R}$ be not open. Then $\bigcup_{n=0}^{\infty} A_n$ is not open.

(c) If x is a limit point of $A \subseteq \mathbb{R}$, then every neighborhood of x contains infinitely many points of A.

(d) If $\lim x_n = 0$ and (y_n) is unbounded, then $\lim x_n y_n$ does not exist.

(e) If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for all $x \in \mathbb{Q}$, then $f = g$.

Problem 3. Let $f_n : A \to \mathbb{R}$ be uniformly continuous functions for $n \in \mathbb{N}$. Suppose that $f_n \to f$ uniformly. Prove that f is uniformly continuous.

Problem 4. Mark each as true or false. Briefly justify your answer.

(a) Every nonempty compact set is uncountable.

(b) For $n \in \mathbb{N}$, let B_n be a finite set. Then $\bigcup_{n=0}^{\infty} B_n$ is countable.

(c) If $A \subseteq \mathbb{Q}$ is connected then A is closed.

(d) If $f : [0,1] \to \mathbb{R}$ is bounded and attains its maximum value, then f attains its minimum value.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $\{x \in \mathbb{R} : f(x) > 0\}$ is open.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x|x^2 + x|$.

(a) Prove using the definition that f differentiable at $x = 0$.

(b) Is f twice differentiable at $x = 0$?

Problem 6. Let

$$f(x) = \sum_{n=0}^{\infty} x^n \cos(nx) = 1 + x\cos(x) + x^2\cos(2x) + \dots.$$

(a) Show that f is continuous on $[-1/2, 1/2]$.

(b) Show that f is differentiable on $(-1/2, 1/2)$.

Problem 7. Let $f : [0,2] \to \mathbb{R}$ be a continuous map and suppose $f(1) < f(0) < f(2)$. Show that there exist $x, y \in (0,2)$ such that $x - y = 1$ and $f(x) = f(y)$.

Problem 8.

(a) Prove that $(x+y)/2 \geq \sqrt{xy}$ for all $x, y > 0$. *[Hint: Start with* $(\sqrt{x} - \sqrt{y})^2 \geq 0$.*]*

(b) Let $a_0, b_0 > 0$ and for $n \geq 0$, define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = (a_n + b_n)/2$. Show that the sequences (a_n) and (b_n) converge.

(c) Show that $\lim a_n = \lim b_n$.

Problem 9. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose that f' is bounded. Show that f is uniformly continuous. *[Hint: Use the mean value theorem.]*