MATH 052: INTRODUCTION TO PROOFS
FINAL EXAM SOLUTIONS

Problem 1. For (a), we have $\{2, 3\}$. For (b), we have "If n is not divisible by 2 and n is not divisible by 5, then n is not divisible by 10." For (c), we have $\mathcal{P}(S) = \{\emptyset, \{-\pi\}, \{\sqrt{2}\}, \{-\pi, -\sqrt{2}\}\}$. Statement (d) is false and (e) is true.

Problem 2. For the truth table, we have:

P	Q	R	$\sim Q \Rightarrow P$	$P \vee R$	$(\sim Q \Rightarrow P) \wedge (P \vee R)$
F	F	F	F	F	F
F	F	T	F	T	F
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

For (b), no, it is not a tautology, because the sentential form is not identically true.

Problem 3. We are given $A \subseteq B$. So to show $A = B$, we show $A \supseteq B$. So let $x \in B$. Since $B \subseteq C$, we have $x \in C$. Since $C \subseteq A$, we have $x \in A$. Thus $B \subseteq A$.

Problem 4. For (a), we have $1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100$.

Now (b). For the base case $n = 1$, we have $1^3 = 1 = 1^2(1 + 1)^2/4$. Now the inductive step: suppose $1^3 + 2^3 + \cdots + n^3 = n^2(n + 1)^2/4$. Then

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2 \frac{n^2 + 4(n+1)}{4} = \frac{(n+1)^2(n+2)^2}{4}. \end{aligned}$$

So by the principle of mathematical induction, the statement is true for all $n \geq 1$.

Problem 5. For (a), we may take $S = \mathcal{P}(\mathbb{R})$, by Cantor's theorem.

For (b), we have the partitions

{1,2,3,4}

{1,2,3},{4} {1,2,4},{3} {1,3,4},{2} {2,3,4},{1}

{1,2},{3,4} {1,3},{2,4} {1,4},{2,3}

{1,2},{3},{4} {1,3},{2},{4} {1,4},{2},{3} {2,3},{1},{4} {2,4},{1},{3} {3,4},{1},{2}

{1},{2},{3},{4}

of which total 15.

For (c), all of (a),(b),(c) need to hold, but not necessarily (d). Finally, for (d), by the binomial theorem, the coefficient is $\binom{63}{60} 2^3$, and optionally we can multiply out:

$$\binom{63}{60} 2^3 = \frac{63 \cdot 62 \cdot 61}{3 \cdot 2} 2^3 = 39711 \cdot 8 = 317688.$$

Problem 6. We show that $g \circ f$ is injective and surjective.

Let $x_1, x_2 \in A$ and suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, this implies that $f(x_1) = f(x_2)$. Since f is injective, this implies $x_1 = x_2$. Thus $g \circ f$ is injective.

Date: 12 December 2011.

Now let $z \in C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Thus $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is surjective.

Problem 7. We show the map is injective and surjective.

Suppose that $f(x_1) = f(x_2)$. Then $9x_1^2 + 5 = 9x_2^2 + 5$, so $x_1^2 = x_2^2$. So $x_1 = \pm x_2$. But $x_1, x_2 \geq 0$, so $x_1 = x_2$. Thus f is injective.

Now let $y \in \mathbb{R}_{\geq 5}$. We find $x \in \mathbb{R}_{\geq 0}$ such that $f(x) = y$. Well, $9x^2 + 5 = y$ so $9x^2 = y - 5$ so $x^2 = (y-5)/9$ and $x = \sqrt{(y-5)/9}$. (Since $y \geq 5$, this square root exists.) Since $f(x) = y$, we have that f is surjective.

Problem 8. We have $a * b = a + b - ab$ and $b * a = b + a - ba = a + b - ab$, so $*$ is commutative. We have

$$a * (b * c) = a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc) = a + b + c - ab - ac - bc + abc$$

and

$$(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)c = a + b + c - ab - ac - bc + abc$$

so $a * (b * c) = (a * b) * c$ and $*$ is associative. The identity element e satisfies $a * e = e * a = a$ for all a:

$$a * e = a + e - ae = a$$

so $e - ae = e(1 - a) = 0$. Since $a \neq 1$, we conclude $e = 0$ is the identity. That finishes (a).

For (b), we have $x \sim x$ since $x - x = 0 \in \mathbb{Z}$, so $\sim$ is reflexive. If $x \sim y$ so $x - y \in \mathbb{Z}$, then $y - x = -(x-y) \in \mathbb{Z}$ so $y \sim x$ and $\sim$ is symmetric. Finally, if $x \sim y$ and $y \sim z$ then $x - y, y - z \in \mathbb{Z}$ so $x - z = (x-y) + (y-z) \in \mathbb{Z}$ so $x \sim z$, and $\sim$ is transitive. Thus $\sim$ is an equivalence relation.

Problem 9. We compute:

$$\begin{aligned} 333 &= 2 \cdot 160 + 13 \\ 160 &= 12 \cdot 13 + 4 \\ 13 &= 3 \cdot 4 + 1 \end{aligned}$$

So $\gcd(333, 160) = 1$ for (a). Thus

$$1 = 13 - 3 \cdot 4 = 13 - 3(160 - 12 \cdot 13) = 37 \cdot 13 - 3 \cdot 160 = 37 \cdot (333 - 2 \cdot 160) - 3 \cdot 160 = 37 \cdot 333 - 77 \cdot 160.$$

So $x = 37$ and $y = -77$.

Problem 10. For (a), suppose $a \mid bc$ and $\gcd(a, c) = 1$. Then there exist $x, y \in \mathbb{Z}$ such that $ax + cy = 1$. Thus

$$b = b \cdot 1 = b(ax + cy) = (ab)x + c(by).$$

Since $c \mid ab$ we have $c \mid (ab)x$. Since also $c \mid c(by)$, we have $c \mid ((ab)x + c(by)) = b$, as desired.

To conclude with (b), suppose that $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$. Then $m \mid (ac - bc) = c(a - b)$. Using (a), we conclude $m \mid (a - b)$. Thus $a \equiv b \pmod{m}$.