MATH 052: INTRODUCTION TO PROOFS
FINAL EXAM

Name ________________________________

Problem	**Score**	**Problem**	**Score**
1		6	
2		7	
3		8	
4		9	
5		10	

Total ____________

Date: 12 December 2011.

Problem 1.

(a) List all elements of the set $\{x \in \mathbb{Z} : x \text{ is prime and } x^2 \leq 23\}$.

(b) Let $n \in \mathbb{Z}$. What is the contrapositive of the statement "If n is divisible by 10, then n is divisible by 2 or divisible by 5"?

(c) What is the power set of $S = \{-\pi, \sqrt{2}\}$?

(d) Label as true or false: $2 \in \{\{1, 2\}\}$ or $\emptyset \in \{1, 2\}$.

(e) Label as true or false: $\sim((\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x \leq y))$.

Problem 2.

(a) Construct a truth table for the following sentential form:

$$(\sim Q \Rightarrow P) \wedge (P \vee R).$$

(b) Is the sentential form in (a) a tautology? Why or why not?

Problem 3. Let A, B, C be sets. Suppose that

$$A \subseteq B \text{ and } B \subseteq C \text{ and } C \subseteq A.$$

Prove (rigorously and carefully) that $A = B$.

Problem 4.

(a) Evaluate $\sum_{k=1}^{4} k^3$.

(b) Prove by induction that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$.

Problem 5.

(a) Give an example of a set S such that $\#S > \#\mathbb{R}$.

(b) How many partitions of $\{1, 2, 3, 4\}$ are there?

(c) Let $S = \{0, 1, 2, 3\}$. Suppose that $\sim$ is a relation on S with $0 \sim 1$ and $2 \sim 1$. Which of the following must also be true if $\sim$ is to be an equivalence relation on S? (Your answer could be none, all or anything in between.)

(a) $3 \sim 3$
(b) $1 \sim 0$
(c) $1 \sim 2$
(d) $2 \sim 3$

(d) What is the coefficient of y^{60} in $(y + 2)^{63}$?

Problem 6. Let $f : A \to B$ and $g : B \to C$ be functions. Show that if f and g are bijections, then $g \circ f$ is a bijection.

Problem 7. Consider the function

$$\begin{aligned} f : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 5} \\ x &\mapsto f(x) = 9x^2 + 5 \end{aligned}$$

(Recall that $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.) Show that f is a bijection.

Problem 8.

(a) Let $S = \mathbb{Q} \setminus \{1\}$. Consider the binary operation $*$ on S given by $a * b = a + b - ab$ for all $a, b \in S$. Show that $*$ is commutative and associative, and determine the identity element for $*$.

(b) Let $S = \mathbb{R}$. Let $\sim$ be the relation on S given by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation.

Problem 9.

(a) Compute $\gcd(333, 160)$ using the Euclidean algorithm.

(b) Compute $x, y \in \mathbb{Z}$ such that $333x + 160y = d$ where $d = \gcd(333, 160)$.

Problem 10.

(a) Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid bc$ and $\gcd(a, c) = 1$. Prove that $a \mid b$.

(b) Now let $a, b, c, m \in \mathbb{Z}$. Suppose that $ac \equiv bc \pmod{m}$ and that $\gcd(c, m) = 1$. Using (a), prove that $a \equiv b \pmod{m}$.