MATH 052: INTRODUCTION TO PROOFS
HOMEWORK #20

Problem 2.9.11(a). Draw a directed graph for each relation in Example 2.53(c) and Example 2.56(b).

Solution. The directed graphs are shown on the next page.

Problem 2.9.11(b)(c)(d). For each of the following, describe a property of $G(R)$ associated with the condition on R.

(b) R is symmetric.
(c) R is reflexive.
(d) R is transitive.

Solution. For (b), R is symmetric if and only if any time there is an edge from one vertex to another, there should be the opposite edge in the other direction.

For (c), R is reflexive if and only if there is a "self-loop" from a vertex to itself for all vertices.

For (d), R is transitive if and only if there is an edge from vertices v to w and an edge from w to x then there is an edge directly from v to x. For example, inExample 2.53(c), R_1 is not transitive but R_2 is.

Problem 2.9.17. Let $\sim$ be an equivalence relation on the set $\{1, 2, 3\}$

(a) Suppose $\sim$ has exactly three ordered pairs. List them explicitly.
(b) Is it possible that $\sim$ has exactly *four* ordered pairs? Explain.
(c) What is the maximum possible number of ordered pairs in $\sim$? When $\sim$ is chosen with this maximum number of ordered pairs, what is the corresponding partition into equivalence classes?

Solution. For (a), since an equivalence relation is reflexive, it must contain the subset $\{(1, 1), (2, 2), (3, 3\}$; if $\sim$ only has three elements, then this must be equality.

For (b), no $\sim$ cannot have four ordered pairs. It must have the three above. If it contains a new ordered pair (a, b) for $a, b \in \{1, 2, 3\}$, then by symmetry it must also contain (b, a), and since these are distinct $\sim$ would have to have at least 5 elements.

Finally, for (c) we can just take $\sim$ to consists of the entire Cartesian product $\{1, 2, 3\} \times \{1, 2, 3\}$, namely, all elements are related to all others. This is an equivalence relation (obviously reflexive, symmetric, and transitive, because it contains all pairs!); the partition of the set consists of a single block containing the entire set.

Problem 2.9.20. For each of the following, determine whether the given relation is or is not an equivalence relation. Then, if it *is* an equivalence relation, describe the partition it induces.

(a) On the set H of human beings, define $x \sim y \Leftrightarrow x$ and y weigh within one pound of each other.
(b) On the set C of all solid-color cars, define $x \sim y \Leftrightarrow x$ and y have the same color.
(c) On the set $\mathbb{N}$ of positive integers, consider the *divisibility* relation defined in Exercise 15(b).
(d) On the set $\mathbb{R}$ of real numbers, define $x \sim y \Leftrightarrow x^2 = y^2$.
(e) On the set $\mathbb{R}$ of real numbers, define $x \sim y \Leftrightarrow xy < 0$.
(f) On the plane $\mathbb{R} \times \mathbb{R}$, define $P \sim Q$ to mean that P and Q have the same y-coordinate.

Date: 14 October 2011.

Solution. For (a), no it is not equivalence relation, it is not transitive. If person x weighs 135 pounds, person y weighs 136 pounds, and person z weighs 137 pounds, then $x \sim y$ and $y \sim z$ but $x \not\sim z$.

For (b), yes, it is an equivalence relation: the equivalence classes consist of cars all having the same color.

For (c), no, it is not symmetric: we have $2 \mid 4$ but $4 \nmid 2$.

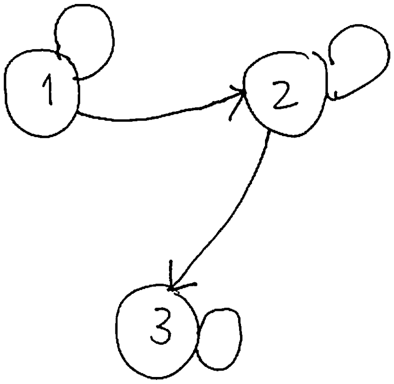
For (d), yes, it is an equivalence relation: the equivalence classes consist of the sets $[0] = \{0\}$ and $[x] = \{x, -x\}$ for $x \neq 0$.

For (e), no, it is not, it is not transitive: if $x = 1$, $y = -1$, and $z = 1$, then $xy = -1 < 0$ and $yz = -1 < 0$ so $x \sim y$ and $y \sim z$, but $xz = 1 > 0$ so $x \not\sim z$.

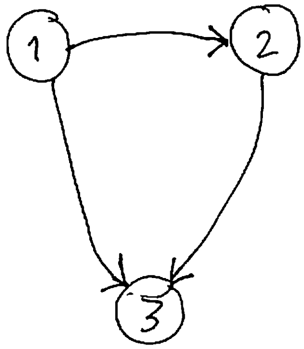
Finally, for (f), it is an equivalence relation: the equivalence classes consist of horizontal lines, $[(x, y)] = \{(a, y) : a \in \mathbb{R}\}$.

2.53(c)

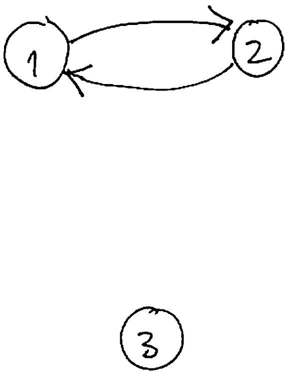
R_1



R_2

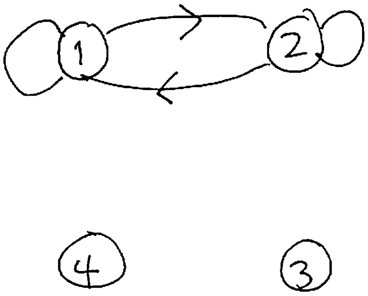


R_3



2.56(b)

R



R_1