MATH 052: INTRODUCTION TO PROOFS
HOMEWORK #26

Problem 3.3.7. Suppose $f : A \to B$ and $g : B \to C$ are functions.

(a) Show that if $g \circ f$ is injective then f is injective.
(b) Show that if $g \circ f$ is surjective then g is surjective.

Solution. First, we prove (a). Suppose that $g \circ f$ is injective; we show that f is injective. To this end, let $x_1, x_2 \in A$ and suppose that $f(x_1) = f(x_2)$. Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$. But since $g \circ f$ is injective, this implies that $x_1 = x_2$. Therefore f is injective.

Next, we prove (b). Suppose that $g \circ f$ is surjective. Let $z \in C$. Then since $g \circ f$ is surjective, there exists $x \in A$ such that $(g \circ f)(x) = g(f(x)) = z$. Therefore if we let $y = f(x) \in B$, then $g(y) = z$. Thus g is surjective.

Problem 3.3.8. In each part of the exercise, give examples of sets A, B, C and functions $f : A \to B$ and $g : B \to C$ satisfying the indicated properties.

(a) g is not injective but $g \circ f$ is injective.
(b) f is not surjective but $g \circ f$ is surjective.

Solution. The same example works for both. Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1\}$, and $f : A \to B$ by $f(1) = 1$ and $g : B \to C$ by $g(1) = g(2) = 1$. Then $g \circ f : A \to C$ is defined by $(g \circ f)(1) = 1$. This map is a bijection from $A = \{1\}$ to $C = \{1\}$, so is injective and surjective. However, g is not injective, since $g(1) = g(2) = 1$, and f is not surjective, since $2 \notin f(A) = \{1\}$.

Problem 3.3.9. Define functions f and g from $\mathbb{Z}$ to $\mathbb{Z}$ such that f is not surjective and yet $g \circ f$ is surjective.

Solution. Let

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto 2n \end{aligned}$$

and

$$\begin{aligned} g : \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto \begin{cases} n/2, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The map f is not surjective: the image is the set of even integers. However, $g \circ f$ is surjective, since $(g \circ f)(n) = g(f(n)) = g(2n)) = (2n)/2 = n$—so in fact $g \circ f$ is the identity map on $\mathbb{Z}$.

Solution. Let

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto \begin{cases} x + 5, & \text{if } x \geq 0; \\ x, & \text{if } x < 0. \end{cases} \end{aligned}$$

Date: 28 October 2011.

and

$$
\begin{aligned}
g : \mathbb{Z} &\to \mathbb{Z} \\
x &\mapsto \begin{cases} x - 5, & \text{if } x \geq 0; \\ x, & \text{if } x < 0. \end{cases}
\end{aligned}
$$

The map f is not surjective: the elements $1, 2, 3, 4$ are not in the image. However, $g \circ f$ is surjective, since $(g \circ f)(x) = g(f(x)) = g(x + 5) = x$ if $x \geq 0$ and $(g \circ f)(x) = f(x) = x$ if $x < 0$—so in fact $g \circ f$ is the identity map on $\mathbb{Z}$.