MATH 052: INTRODUCTION TO PROOFS
HOMEWORK #32

Problem 5.8.1(a)(b)(c). Compute the following binomial coefficients.

(a) $\binom{9863}{1}$
(b) $\binom{805}{2}$
(c) $\binom{14}{4}$

Solution. For (a), we have $\binom{9863}{1} = 9863$, since $\binom{n}{1} = n$ for all $n \geq 1$. For (b), we have $\binom{805}{2} = 805(804)/2 = 323610$. For (c), we have

$$\binom{14}{4} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2} = 13 \cdot 11 \cdot 7 = 1001.$$

Problem 5.8.4. Determine all the nonnegative integers n that satisfy the equation

$$\binom{n}{4} = \binom{n}{6}.$$

Solution.

$$\begin{aligned} \binom{n}{4} &= \binom{n}{6} \\ \frac{n(n-1)(n-2)(n-3)}{24} &= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720}. \end{aligned}$$

The nonnegative integers $n = 0, 1, 2, 3$ are solutions to the equation, since for these n we have $\binom{n}{4} = \binom{n}{6} = 0$. Cancelling these from both sides, we obtain

$$\begin{aligned} 30 &= (n-4)(n-5) \\ n^2 - 9n - 10 &= 0 \\ n &= -1, 10. \end{aligned}$$

Since $n = -1$ is negative, all nonnegative values of n that satisfy the equation are $n = 0, 1, 2, 3, 10$.

Problem 5.8.5. A local restaurant offers a special salad lunch subject to these conditions: Each diner chooses exactly five ingredients from a menu of eight possibilities and then tops it with a blend of exactly two of the restaurant's four dressings.

(a) How many dressed salad combinations are possible?
(b) Jones loves the lunch, but he cannot tolerate the simultaneous inclusion of radishes (one of the eight items on the menu) and peanut butter dressing (one of the four hoices), though all other possibilities are acceptable to him. How many acceptable choices does he have?
(c) In the statement of the lunch conditions, change each "exactly" to "at most." Now answer (a) again. (Assume that the diner chooses at least one salad ingredient, but allow for the possibility of no dressing.)

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Solution. For (a), there are $\binom{8}{5}$ choices for ingredients and $\binom{4}{2}$ choices for dressings, so a total of

$$\binom{8}{5}\binom{4}{2} = 56 \cdot 6 = 336$$

dressed salad combinations.

For (b), among the 336 possibilities, we remove those that have both the choice of radishes and peanut butter dressing. This leaves Jones $\binom{7}{4} = 35$ choices for ingredients and $\binom{3}{1} = 3$ choices for dressing, so the excluded possibilities are $35 \cdot 3 = 105$. The total remaining choices are $336 - 105 = 231$.

For (c), for the toppings, we may choose a total of 1 through 5 total number of items, giving

$$\binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} = 8 + 28 + 56 + 70 + 56 = 218.$$

possibilities. For the dressings, we have

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1 + 4 + 6 = 11$$

choices, for a total of $218 \cdot 11 = 2398$ possibilities.

Problem 5.8.8. What is the coefficient of y^3 in the expansion of $(2x - 5y)^6$?

Solution. We use the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

We let $a = 2x$ and $b = -5y$; we have $n = 6$ so since the power of y is $n - k = 3$ we have $k = 3$. Thus the term that appears is

$$\binom{6}{3}(2x)^3(-5y)^3 = 20(8x^3)(-125y^3) = (-20000x^3)y^3.$$

The coefficient of y^3 is $-20000x^3$.