MATH 052: INTRODUCTION TO PROOFS
HOMEWORK #37

Problem 6.3.8. Define a *prime triple* to be a set of three prime numbers of the form $\{n, n+2, n+4\}$. For example, $\{3, 5, 7\}$ is a prime triple. Are there any others? Either exhibit another or prove there are none.

Solution. We make the following claim.

Claim. If $n \in \mathbb{Z}$ then at least one of $n, n + 2, n + 4$ is divisible by 3.

Proof. By the division algorithm, we can write $n = 3q + r$ with $r = 0, 1, 2$. If $r = 0$, then $n = 3q$ is divisible by 3; if $r = 1$, then $n + 2 = 3(q + 1)$ is divisible by 3; if $r = 2$, then $n + 4 = 3(q + 2)$ is divisible by 3. ☐

Now if $\{n, n + 2, n + 4\}$ is a set of three prime numbers, then by the claim one of them must be divisible by 3 and hence equal to 3 since it is prime. Thus either $n = 3$, in which case we have the triple $\{3, 5, 7\}$; or $n + 2 = 3$, so we obtain the triple $\{1, 3, 5\}$ but 1 is not prime; or $n + 4 = 3$, but then $n = -1 < 0$. So the only prime triple is $\{3, 5, 7\}$.

Problem 6.3.10(a)(b). Use Euclid's algorithm to compute the following greatest common divisors $\gcd(a, b)$ and the extended Euclidean algorithm to write $\gcd(a, b)$ as a linear combination of a, b.

(a) $\gcd(56, 104)$
(b) $\gcd(462, 3003)$

Solution. For (a), we have:

$$\begin{aligned} 104 &= 1 \cdot 56 + 48 \\ 56 &= 1 \cdot 48 + 8 \\ 48 &= 6 \cdot 8 \end{aligned}$$

Therefore $\gcd(104, 56) = 8$. The extended Euclidean algorithm gives:

$$8 = 1 \cdot 56 + (-1) \cdot 48 = 1 \cdot 56 + (-1)(104 - 1 \cdot 56) = (-1)(104) + 2(56).$$

In a similar way, we obtain $\gcd(462, 3003) = 231$ with $231 = -6(462) + 1(3003)$.

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