MATH 052: FUNDAMENTALS OF MATHEMATICS
EXAM #2

Problem 1. For (a), the two distributive laws are:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

For (b), a counterexample is $n = 3$: we have $3^2 = 9 \equiv 1 \pmod{4}$, since $4 \mid (9-1) = 8$, but $3 \not\equiv 1 \pmod{4}$, since $4 \nmid (3-1) = 2$. For (c), the statement that is proved is:

If $a \equiv 2 \pmod{4}$ and $b \equiv 1 \pmod{4}$, then $4 \nmid (a^2 + 2b)$.

Problem 2. For (a), the statement is true: we can take $a = 2$ and $b = -1$ and get $ab = -2 < 0$ and $a + b = 2 - 1 = 1 > 0$. For (b), we write the statement as

$$\exists a, b \in \mathbb{Z}, (ab < 0) \wedge (a + b > 0).$$

For (c), the negation is

$$\forall a, b \in \mathbb{Z}, (ab \geq 0) \vee (a + b \leq 0).$$

For (d), this reads

For all integers a and b, we have either $ab \geq 0$ or $a + b \leq 0$.

Problem 3. Let $a, b \in \mathbb{Z}$. We prove the contrapositive: if a is even and b is even then $ax+by$ is even. Since a is even and b is even, we have $a = 2m$ and $b = 2n$ for $m, n \in \mathbb{Z}$. Thus $ax+by = (2m)x+(2n)y = 2(mx+ny)$, so $ax + by$ is even.

Problem 4. Since $a \mid b$, by definition there exists $x \in \mathbb{Z}$ such that $b = xa$. Similarly, since $b \mid c$, there exists $y \in \mathbb{Z}$ such that $c = yb$. Therefore $c = yb = y(xa) = (xy)a$, and since $xy \in \mathbb{Z}$, we have by definition that $a \mid c$.

Problem 5. First, we prove the base case ($n = 1$): we verify indeed that

$$1 = \frac{1-r}{1-r} = 1.$$

Now, the induction step. Assume that

$$1 + r + r^2 + \cdots + r^{k-1} = \frac{1 - r^k}{1 - r}.$$

We want to show that

$$1 + r + r^2 + \cdots + r^{k-1} + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

Well, by the induction hypothesis, we have

$$1 + r + r^2 + \cdots + r^{k-1} + r^k = \frac{1 - r^k}{1 - r} + r^k = \frac{1 - r^k}{1 - r} + \frac{r^k(1-r)}{1 - r} = \frac{1 - r^k + r^k - r^{k+1}}{1 - r} = \frac{1 - r^{k+1}}{1 - r}.$$

Therefore, by the principle of mathematical induction, the result holds for all $n \geq 1$.

Date: 31 October 2012.