MATH 052: FUNDAMENTALS OF MATHEMATICS
EXAM #2

Name ____

Problem	**Score**
1	
2	
3	
4	
5	

Total ______________

Date: 31 October 2012.

Problem 1.

(a) For sets A, B, the commutative laws state that $A \cup B = B \cup A$ and $A \cap B = B \cap A$. State the two distributive laws for sets.

(b) Disprove the statement: If $n \in \mathbb{Z}$ and $n^2 \equiv 1 \pmod 4$ then $n \equiv 1 \pmod 4$.

(c) A proof of a result is given below. What result is proved?

Proof. Let $a \equiv 2 \pmod 4$ and $b \equiv 1 \pmod 4$ and assume, to the contrary, that $4 \mid (a^2 + 2b)$. We have $a = 4r + 2$ and $b = 2s + 1$ with $r, s \in \mathbb{Z}$. Therefore,

$$a^2 + 2b = (4r + 2)^2 + 2(2s + 1) = 16r^2 + 16r + 4s + 6.$$

Now since $4 \mid (a^2 + 2b)$, we have $a^2 + 2b = 4t$ with $t \in \mathbb{Z}$. So $16r^2 + 16r + 4s + 6 = 4t$ hence

$$6 = 4t - 16r^2 - 16r - 4s = 4(t - 4r^2 - 4r - s).$$

Since $t - 4r^2 - 4r - s \in \mathbb{Z}$, we conclude that $4 \mid 6$, which is a contradiction. □

Problem 2. The statement

For every integer m, either $m \leq 1$ or $m^2 \geq 4$

can be expressed using quantifiers as

$$\forall m \in \mathbb{Z}, (m \leq 1) \vee (m^2 \geq 4).$$

Consider the statement

There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

(a) Determine the truth value of this statement.

(b) Express this statement using quantifiers.

(c) Express using quantifiers the negation of this statement.

(d) Express in words the negation of this statement.

Problem 3. Let $x, y, a, b \in \mathbb{Z}$. Prove that if $ax + by$ is odd then a is odd or b is odd.

Problem 4. Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid c$ then $a \mid c$.

Problem 5. Let $r \in \mathbb{R}$ satisfy $r \neq 1$. Prove by induction that

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

for all integers $n \geq 1$.