# MATH 250B: COMMUTATIVE ALGEBRA
# HOMEWORK #3

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Some solutions are omitted or sketched.

**Problem 6**. *Let $M, N$ be flat. Show that $M \otimes N$ is flat.*

*Solution.* If $E' \hookrightarrow E$ is injective, then since $M$ is flat, so is the map $E' \otimes M \hookrightarrow E \otimes M$; since $N$ is also flat, so too is the map $(E' \otimes M) \otimes N \hookrightarrow (E \otimes M) \otimes N$. By associativity of the tensor product, we obtain the injection

$$E' \otimes (M \otimes N) \hookrightarrow E \otimes (M \otimes N),$$

which is to say, $M \otimes N$ is flat.

**Problem 7**. *Let $F$ be a flat $R$-module, and let $a \in R$ be an element which is not a zerodivisor. Show that if $ax = 0$ for some $x \in F$ then $x = 0$.*

*Solution.* The map $R \xrightarrow{a} R$ which is multiplication by $a$ is injective since $a$ is not a zerodivisor, by definition. Since $M$ is flat, the map $R \otimes F \xrightarrow{a \otimes 1} R \otimes F$ is also injective. Since $R \otimes F \cong F$, we see that the map $F \xrightarrow{a} F$ is injective, which is what we were to show.

**Problem 8**. *Prove the following:*

(i) *Let $S$ be a multiplicative subset of $R$. Then $S^{-1}R$ is flat over $R$.*
(ii) *A module $M$ is flat over $R$ if and only if the localization $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of $R$.*
(iii) *Let $R$ be a principal (entire) ring. A module $F$ is flat if and only if $F$ is torsion free.*

*Solution.* Statement (i) follows from the fact that $S^{-1}E \cong S^{-1}R \otimes_R E$, and if $E' \to E$ is injective, then so is $S^{-1}E' \to S^{-1}E$.

For (ii), note that $M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M$, so taking $S = R \setminus \mathfrak{p}$, we may apply (i) and Problem 6 to see that $M_\mathfrak{p}$ is flat for every prime ideal $\mathfrak{p}$. For the converse, if $M$ is not flat, then there is an injection $E' \to E$ of $R$-modules such that $E' \otimes M \to E \otimes M$ is no longer an injection; let $N$ be the kernel of this map. There exists a prime $\mathfrak{p}$ such that $N_\mathfrak{p} \neq 0$ (for example, choose a maximal ideal containing all elements which annihilate $N$). Then $M_\mathfrak{p}$ is clearly not flat.

For (iii), one direction follows from Problem 7, and for the converse, if $F$ is torsion free then since $R$ is a principal entire ring, $F$ is the direct limit of its finitely generated submodules which are free by III.7.3, hence flat (using Exercise 12 below).

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**Problem 9**. *Let $M$ be an $A$-module. We say that $M$ is* faithfully flat *if $M$ is flat, and if the functor*

$$T_M : E \mapsto M \otimes_A E$$

*is faithful, that is $E \neq 0$ implies $M \otimes_A E \neq 0$. Prove that the following conditions are equivalent:*

(i) *$M$ is faithfully flat;*
(ii) *$M$ is flat, and if $u : F \to E$ is a homomorphism of $A$-modules, $u \neq 0$, then $T_M(u) : M \otimes_A F \to M \otimes_A E$ is also nonzero;*
(iii) *$M$ is flat, and for all maximal ideals $\mathfrak{m}$ of $A$, we have $\mathfrak{m}M \neq M$; and*
(iv) *A sequence of $A$-modules $N' \to N \to N''$ is exact if and only if the sequence tensored with $M$ is exact.*

*Solution.* First, we show (i) $\Rightarrow$ (ii). Let $E'$ be the image of $u$. Since $u \neq 0$ we know $E' \neq 0$. Then the image of $T_M(u)$ is $M \otimes_A E'$ which is nonzero, since $M$ is faithful.

For (ii) $\Rightarrow$ (iii), take the nonzero homomorphism $u : A \to A/\mathfrak{m}A$; we have $T_M(u) : M \to M/\mathfrak{m}M$, so $M/\mathfrak{m}M \neq 0$, i.e. $\mathfrak{m}M \neq M$.

Now we show (iii) $\Rightarrow$ (i). Suppose that $E \neq 0$; let $x \in E$ be a nonzero element, and let $\mathfrak{m}$ be a maximal ideal containing the annihilator of $x$; then we have an injection of $R$-modules

$$x(R/\mathfrak{m}) \hookrightarrow E/\mathfrak{m}E.$$

(If $ax = 0$ then $a$ is in the annihilator of $x$.) Tensoring with $M$, we obtain

$$x(R/\mathfrak{m}) \otimes M \hookrightarrow (E/\mathfrak{m}E) \otimes M$$

which becomes

$$x(M/\mathfrak{m}M) \hookrightarrow (E \otimes M) \otimes R/\mathfrak{m}.$$

By (iii), $M/\mathfrak{m}M \neq 0$, so $E \otimes M \neq 0$.

To show (iv) $\Rightarrow$ (i), taking $N' = 0$ we see that $M$ is flat; if $E \otimes M = 0$ then $0 \otimes M \to E \otimes M \to 0 \otimes M$ is exact, so $0 \to E \to 0$ is exact, so $E = 0$.

To conclude, we show that (i) $\Rightarrow$ (iv). If $M$ is flat, then $N' \to N \to N''$ exact implies $N' \otimes M \to N \otimes M \to N'' \otimes M$ exact. Conversely, if

$$N' \otimes M \xrightarrow{f\otimes 1} N \otimes M \xrightarrow{g \otimes 1} N'' \otimes M$$

is exact, then $(\operatorname{img} g / \ker f) \otimes M = 0$, so by (i) we have $\operatorname{img} g/\ker f = 0$, i.e.

$$N' \xrightarrow{f} N \xrightarrow{g} N''$$

is exact.

**Problem 12**. *Show that the tensor product commutes with direct limits. In other words, if $\{E_i\}$ is a direct family of modules, and $M$ is any module, then there is a natural isomorphism*

$$\varinjlim (E_i \otimes_A M) \cong (\varinjlim E_i) \otimes_A M.$$

*Solution (sketch).* Show that $(\varinjlim E_i) \otimes_A M$ satisfies the universal property of the direct limit; this is clear on any finite level.

**Problem ***. *Let $R = k[x, y]$ be the indicated polynomial ring in two variables over a field $k$. Show that the maximal ideal $(x, y)$ of $R$ is not flat over $R$.*

*Solution.* Using Proposition 3.7, we note that it suffices to show that the multiplication map $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}^2$ is not an isomorphism. This follows because

$$x \otimes y - y \otimes x = xy - yx = 0,$$

so the map even fails to be injective.

Note that $x \otimes y - y \otimes x \neq 0$, because there is a $R$-linear map

$$\begin{aligned} f : \mathfrak{m} \times \mathfrak{m} &\to k[x, y]/(x, y) \\ (a, b) &\mapsto (\partial a/\partial x)(\partial b/\partial y) \end{aligned}$$

with $f(x, y) = 1 \neq 0 = f(y, x)$.