MATH 250B: COMMUTATIVE ALGEBRA
HOMEWORK #5

JOHN VOIGHT

Problem 1. *Let A be a commutative ring. Let M be a module, and N a submodule. Let $N = Q_1 \cap \dots \cap Q_r$ be a primary decomposition of N. Let $\overline{Q_i} = Q_i/N$. Show that $0 = \overline{Q_1} \cap \dots \cap \overline{Q_r}$ is a primary decomposition of 0 in M/N. State and prove the converse.*

Solution. Set (or module) theoretically, we indeed have $0 = \bigcap_i \overline{Q_i}$ if and only if $N = \bigcap_i Q_i$.

Now $Q \subset M$ is primary if and only if $Q \neq M$ and $a_{M/Q}$ is injective or nilpotent. Therefore $\overline{Q} = Q/N \subset M/N$ is primary if and only if $Q/N \neq M/N$ and $a_{(M/N)/(Q/N)}$ is injective or nilpotent. For any $Q \supset N$, we have $(M/N)/(Q/N) \cong M/Q$, so $Q \subset M$ is primary (contaning N) if and only if $\overline{Q} \subset M/N$ is primary.

The converse is also now proven: Let $0 = \bigcap_i \overline{Q_i}$ be a primary decomposition for 0 in M/N. Let Q_i be the inverse image of $\overline{Q_i}$ in the surjection $M \to M/N$. Then $N = \bigcap_i \overline{Q_i}$ is a primary decomposition for N.

Problem 2. *Let $\mathfrak{p}$ be a prime ideal, and $\mathfrak{a}, \mathfrak{b}$ ideals of A. If $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, show that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.*

Solution. If $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$, then there exists an $a \in \mathfrak{a} \setminus \mathfrak{p}$ and $b \in \mathfrak{b} \setminus \mathfrak{p}$. Then $ab \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, so since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, a contradiction.

Problem 3. *Let $\mathfrak{q}$ be a primary ideal. Let $\mathfrak{a}, \mathfrak{b}$ be ideals, and assume $\mathfrak{a}\mathfrak{b} \subset \mathfrak{q}$. Assume that $\mathfrak{b}$ is finitely generated. Show that $\mathfrak{a} \subset \mathfrak{q}$ or there exists some positive integer n such that $\mathfrak{b}^n \subset \mathfrak{q}$.*

Solution. If $\mathfrak{a} \subset \mathfrak{q}$, we are done. Otherwise, let $a \in \mathfrak{a} \setminus \mathfrak{q}$. Let $b_1, \dots, b_r$ generate $\mathfrak{b}$ as an ideal. Then for each i, $ab_i \in \mathfrak{q}$ but $a \notin \mathfrak{q}$ so there exists an integer n_i such that $b_i^{n_i} \in \mathfrak{q}$. Let $n = r \max_i n_i$. Then $\mathfrak{b}^n$ is generated by products $b = b_1^{m_1} \dots b_r^{m_r}$ with $\sum_i m_i = n$; by the pigeonhole principle i such that $e_i \geq n_i$, so $b \in \mathfrak{q}$ for all $b \in \mathfrak{b}^n$.

Problem 4. *Let A be noetherian, and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Show that there exists some $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$.*

Date: March 6, 2003.

X: 1–8, *.

Solution. Since A is noetherian, the ideal $\mathfrak{p}$ is finitely generated; let $p_1, \dots, p_r$ generate $\mathfrak{p}$. We have by definition that

$$\mathfrak{p} = \{a \in A : a^n \in \mathfrak{q} \text{ for some } n \geq 1\}.$$

Let n_i be such that $p_i^{n_i} \in \mathfrak{q}$, and let $n = r \max_i n_i$. Then as in the previous exercise we see that $\mathfrak{p}^{rn} \subset \mathfrak{q}$.

Problem 5. *Let A be an arbitrary commutative ring and let S be a multiplicative subset. Let $\mathfrak{p}$ be a prime ideal and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Then $\mathfrak{p}$ intersects S if and only if $\mathfrak{q}$ intersects S. Furthermore, if $\mathfrak{q}$ does not intersect S, then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary in $S^{-1}A$.*

Solution. If $f \in \mathfrak{p} \cap S$, then there exists an n such that $f^n \in \mathfrak{q}$ since $\mathfrak{q}$ is $\mathfrak{p}$-primary and S is multiplicatively closed. The reverse inclusion follows immediately from $\mathfrak{q} \subset \mathfrak{p}$.

To the second statement, we know that $S^{-1}\mathfrak{q} \neq A$, since $S \cap \mathfrak{q} = \emptyset$. We now see that

$$S^{-1}A/S^{-1}\mathfrak{q} \cong S^{-1}(A/\mathfrak{q}),$$

since the map $S^{-1}A \to S^{-1}(A/\mathfrak{q})$ clearly has kernel $S^{-1}\mathfrak{q}$. Therefore $a_{S^{-1}A/S^{-1}\mathfrak{q}} = a_{S^{-1}(A/\mathfrak{q})}$, we know that if $a_{A/\mathfrak{q}}$ is injective or nilpotent then $a_{S^{-1}(A/\mathfrak{q})}$ is injective (since localization is flat) or nilpotent, respectively.

Finally, we see that $a \in A$ has $a^n \in \mathfrak{q}$ for some $n \geq 1$ if and only if $a/1 \in S^{-1}A$ has $(a/1)^n \in S^{-1}\mathfrak{q}$ if and only if for all $s \in S$, $(a/s)^n \in S^{-1}\mathfrak{q}$, so $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary. (Or note that $S^{-1}\mathfrak{p} = S^{-1}\sqrt{\mathfrak{q}} = \sqrt{S^{-1}\mathfrak{q}}$.)

Problem 6. *If $\mathfrak{a}$ is an ideal of A, show there is a bijection between the prime ideals of A which do not intersect S and the prime ideals of $S^{-1}A$, given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{p} \mapsto S^{-1}\mathfrak{p} \cap A = \mathfrak{p}$.*

Prove a similar statement for primary ideals instead of prime ideals.

Solution (sketch). Let $\mathfrak{p} \subset A$ be a prime ideal. We show that $\mathfrak{p} \cap S = \emptyset$ if and only if $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$. From the previous exercise, we have $S^{-1}(A/\mathfrak{p}) \cong S^{-1}A/S^{-1}\mathfrak{p}$. Since $A/\mathfrak{p}$ is a domain, $S^{-1}(A/\mathfrak{p})$ is a domain if and only if $S \cap \mathfrak{p} = \emptyset$ if and only if $S^{-1}\mathfrak{p}$ is prime.

Also from the previous exercise, we see that $S^{-1}\mathfrak{q}$ is primary if and only if $S^{-1}\mathfrak{p}$ is primary, and by the correspondence of primes we have a similar bijection for $\mathfrak{p}$-primary ideals.

Problem 7. *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$ be a reduced primary decomposition of an ideal. Assume that $\mathfrak{q}_1, \dots, \mathfrak{q}_i$ do not intersect S, but that $\mathfrak{q}_j$ intersects S for $j > i$. Show that*

$$S^{-1}\mathfrak{a} = S^{-1}\mathfrak{q}_1 \cap \dots \cap S^{-1}\mathfrak{q}_i$$

is a reduced primary decomposition of $S^{-1}\mathfrak{a}$.

Solution. Note that for $j > i$, since $\mathfrak{q}_j \cap S = \emptyset$, we have $S^{-1}\mathfrak{q}_j = S^{-1}A$, so already $S^{-1}\mathfrak{a} = \bigcap_{j=1}^{i} S^{-1}\mathfrak{q}_j$ is a primary decomposition of $S^{-1}\mathfrak{a}$.

The bijection of the previous exercise implies that the primes $S^{-1}\mathfrak{p}_i$ are distinct whenever $\mathfrak{p}_i$ are distinct, for $j \leq i$. If the decomposition is not reduced, then we have

$$S^{-1}\mathfrak{q}_k \subset \bigcap_{j \neq k} S^{-1}\mathfrak{q}_j$$

for some $1 \leq i \leq k$; as the bijection preserves inclusions, we have

$$S^{-1}\mathfrak{q}_k \cap A \subset \mathfrak{q} \bigcap_{j \neq k} S^{-1}\mathfrak{q}_j \cap A = \bigcap_{j \neq k} \mathfrak{q}_j$$

so the original decomposition is not reduced, a contradiction.

Problem 8. *Let A be a local ring. Show that any idempotent $\neq 0$ in A is necessarily the unit element.*

Solution. Note that $e^2 = e$ if and only if $(1-e)^2 = 1-e$. We cannot have both e and $1-e$ in the maximal ideal $\mathfrak{m}$, since then $e + (1-e) = 1 \in \mathfrak{m}$. Therefore, say, $e \notin \mathfrak{m}$, so since A is local, e is a unit. But $e(1-e) = 0$, so $1 - e = 0$, i.e. $e = 1$.

Problem *. *Find an example of an ideal I in an integral domain A for which there is a prime in $\operatorname{Ass}(A/I)$ that is not in $\operatorname{Ass}(A)$.*

Solution. Take $I = p^2\mathbb{Z}$ and $A = \mathbb{Z}$. Then $\operatorname{Ass}(\mathbb{Z}) = \{(0)\}$ since the annihilator of any $x \in \mathbb{Z}$ is just 0. However, $\operatorname{Ass}(\mathbb{Z}/p^2\mathbb{Z}) = \{(0), (p)\}$, since this is the set of all primes of $\mathbb{Z}/p^2\mathbb{Z}$ and (0) is the annihilator of 1 and (p) is the annihilator of the element $p \in \mathbb{Z}/p^2\mathbb{Z}$.