# MATH 250B: COMMUTATIVE ALGEBRA
# HOMEWORK #6

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**Problem 1**. *Let $R$ be a ring. Let $N = \operatorname{rad} R = \bigcap_{\mathfrak{m}} \mathfrak{m}$ be the intersection of all maximal left ideals of $R$.*

(a) *Show that $NE = 0$ for every simple $R$-module $E$. Show that $N$ is a two-sided ideal.*
(b) *Show that* $\operatorname{rad}(R/N) = 0$.

*Solution.* By the beginning of (XVII §6), there is a bijection between maximal left ideals and simple $R$-modules (up to isomorphism) and $E \cong R/M$ for a maximal left ideal $M$. Already, then, $N \subset M = \operatorname{Ann} E$, so $NE = 0$. Moreover, since $N(R/M) = 0$ for all maximal left ideals $M$, $NR \subset M$ for all $M$ and therefore $NR \subset \bigcap_M M = N$, so $N$ is also a right ideal.

For (b), note that every maximal ideal of $R/N$ is the image of a maximal ideal $M$ of $R$ containing $N$ (under the natural surjection $R \to R/N$). But every maximal ideal contains $N$, so $\operatorname{rad}(R/N)$ is the image of $\operatorname{rad}(R) = N$ in $R/N$, i.e. $\operatorname{rad}(R/N) = 0$.

**Problem 2**. *A ring is* (left) artinian *if every descending sequence of left ideals stabilizes.*

(a) *Show that a finite-dimensional algebra $A$ over a field $k$ is artinian.*
(b) *If $R$ is artinian, show that every nonzero left ideal contains a simple left ideal.*
(c) *If $R$ is artinian, show that every nonempty set of left ideals $S$ contains a minimal left ideal.*

*Solution.* A left ideal of $A$ is in particular a finite-dimensional $k$-vector space, so for any sequence $J_1 \supset J_2 \supset \dots$ we have $\dim_k A \geq \dim_k J_1 \geq \dots$ and this can only have finitely many strict inequalities; therefore the original sequence of ideals stabilizes.

For (b), let $J = J_1$ be any nonzero left ideal of $R$. If $J$ itself is simple, we are done; otherwise it properly contains a nonzero left ideal $J_2$. Continuing in this fashion, we obtain $J = J_1 \supsetneq J_2 \supsetneq \dots$ which stabilizes as $R$ is artinian. The final stable factor $J_n \neq 0$ is a simple left ideal of $J$.

Finally, for (c) let $J_1 \in S$; if $J_1$ is not minimal, there exists $J_2 \in S$ such that $J_1 \supsetneq J_2$. Continuing in this way we obtain a descending chain of left ideals and since $R$ is artinian, this procedure must eventually terminate and $J_n$ is a minimal ideal in $S$.

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XVII: 1–7, 9, 10, 13.

**Problem 3**. *Let $R$ be artinian. Show that* $\operatorname{rad} R = 0$ *if and only if $R$ is semisimple.*

*Solution.* We must assume that $R$ is not the zero ring since by definition the zero ring is not semisimple (§4).

Since $R$ is artinian, we claim that we can write $\operatorname{rad} R = \bigcap_{i=1}^t M_i$ for a finite set of maximal (left) ideals $M_1, \dots, M_t$. Otherwise, there would exist a descending chain $M_1 \supsetneq M_1 \cap M_2 \supsetneq \dots$ of left ideals of $R$.

Suppose that $\operatorname{rad} R = 0$. Then the map $R \to \bigoplus_i R/M_i$ is injective, since its kernel is $\bigcap_i M_i = \operatorname{rad} R = 0$. But each $R/M_i$ is simple, since each $M_i$ is maximal; therefore $R$ is (isomorphic to) a submodule of a semisimple module, so $R$ is semisimple.

If $R$ is semisimple, write $R \cong \bigoplus_i E_i$ an isomorphism of $R$-modules with $E_i$ simple. Then $M_i = \operatorname{Ann} E_i$ is a maximal left ideal of $R$ and $\bigcap_i \operatorname{Ann} E_i = \bigcap_i M_i = 0$. But $\operatorname{rad} R \subset \bigcap_i M_i = 0$, so $\operatorname{rad} R = 0$. Note this does not require that $R$ be artinian!

**Problem 4 (Nakayama's Lemma)**. *Let $R$ be a ring and $M$ a finitely generated module. Let $N = \operatorname{rad} R$. If $NM = M$, show that $M = 0$.*

*Solution.* Let $m_1, \dots, m_r$ be a minimal generating set for $M$ (i.e. no smaller subset generates $M$). Then there exist $n_1, \dots, n_r \in N$ such that $n_1 m_1 + \dots + n_r m_r = m_r$, so $(1 - n_r)m_r = n_1 m_1 + \dots + n_{r-1} m_{r-1}$. We know $1 - n_1$ must be left invertible (otherwise $1 - n_1$ is contained in some maximal left ideal and so too would $n_1 + (1 - n_1) = 1$)—but then $m_1, \dots, m_{r-1}$ generate $M$, contradicting minimality. Therefore $M$ had no such minimal generating set, i.e. $M = 0$.

**Problem 5**.

(a) *Let $J$ be a two-sided nilpotent ideal of $R$. Show that $J$ is contained in the (Jacobson) radical.*

(b) *Conversely, assume that $R$ is Artinian. Show that its Jacobson radical is nilpotent, i.e., that there exists an integer $r \geq 1$ such that $N^r = 0$. [Hint: Consider the descending sequence of powers $N^r$, and apply Nakayama to a minimal finitely generated left ideal $L \subset N^\infty$ such that $N^\infty L \neq 0$.]*

*Solution.* For (a), suppose that $x \in J$ has $x \notin M$ for some maximal ideal $M$. Then $Rx + M = R$, so there exists an $a \in R$ and $m \in M$ such that $ax + m = 1$. But $ax \in J$ is nilpotent, so say $(ax)^n = 0$. Then

$$(1 + \dots + (ax)^{n-1})(1 - ax) = 1 - (ax)^n = 1$$

so $m = 1 - ax$ is left invertible, a contradiction since $m \in M$. Hence no such $x$ exists, and $J \subset \operatorname{rad} R$.

For (b), let $N = \operatorname{rad} R$; then $N \supset N^2 \supset \dots$ is a descending chain of left ideals, so it stabilizes, say the limit is $N^r = N^{r+1} = \dots$. Suppose that $N^r \neq 0$: then in particular, $N^r N = N^{r+1} = N^r \neq 0$. Let $S$ be the set of left ideals $L$ such that $N^r L \neq 0$. We have shown that $N \in S$, so $S \neq \emptyset$, so by Exercise 2(c), there is a minimal element $L \in S$. By minimality, we know that $L$ is finitely generated (if $N^r L \neq 0$ there exists $x \in L$ such that $N^r x \neq 0$, so $Rx \in S$ and hence $L = Rx$). By Nakayama's lemma, since $N(N^r L) = N^r L$, we must have $N^r L = 0$, a contradiction. Therefore $N^r = 0$.

**Problem 6**. *Let $R$ be a semisimple commutative ring. Show that $R$ is a direct product of fields.*

*Solution.* By Theorem 4.3, every semisimple ring is the (finite) product of simple rings. A commutative simple ring $k$ must be a field: the zero ring is not simple, and if $a \in k$ is nonzero, then $(a)$ is a (two-sided) ideal hence by Theorem 5.2, $(a) = k$, so $a \in k^*$ and $k$ is a field.

**Problem 7**. *Let $R$ be a finite-dimensional commutative algebra over a field $k$. If $R$ has no nilpotent element $\neq 0$, show that $R$ is semisimple.*

*Solution.* $R$ is Artinian by Exercise 2(a), and $\operatorname{rad} R$ is nilpotent by Exercise 5(b); by hypothesis, then, $\operatorname{rad} R = 0$, so by Exercise 3, $R$ is semisimple.

**Problem 10**. *Let $E$ be a finite-dimensional vector space over a field $k$. Let $A \in \operatorname{End}_k(E)$. Show that the $k$-algebra $R$ generated by $A$ is semisimple if and only if its minimal polynomial has no factors of multiplicity $> 1$ over $k$.*

*Solution.* Let $m(t) = \prod_{i=1}^{r} p_i(t)^{e_i}$ be the minimal polynomial of $A$ written as the product of irreducibles. Then by the Chinese remainder theorem,

$$R = k[t]/(m(t)) \cong \prod_{i=1}^{r} k[t]/(p_i(t)^{e_i}).$$

The $k$-algebra $R$ is commutative, so by Exercise 6, $R$ is semisimple (if and) only if $R$ is a direct product of fields. But $R$ is a product of fields if and only if $e_i = 1$ for each $i$, for otherwise one factor has a nilpotent element, which is the result.