MATH 250B: COMMUTATIVE ALGEBRA HOMEWORK #7

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Problem XVII.12. *Prove that an R-module E is a generator if and only if it is balanced and finitely generated projective over* $\operatorname{End}_R E$*.*

Solution. Lang proves the ($\Rightarrow$) direction as Theorem 7.1, so it suffices to show that if E is balanced and finitely generated projective over $\operatorname{End}_R E$, then R is a homomorphic image of a direct sum of E with itself.

Since E is finitely generated projective over $\operatorname{End}_R E$, we have an isomorphism $(\operatorname{End}_R E)^n \cong E \oplus F$ for some $\operatorname{End}_R E$-module F. Therefore we have the following isomorphisms of $\operatorname{End}_R E$-modules:

$$\begin{aligned} E^n &\cong \operatorname{Hom}_{\operatorname{End}_R E}((\operatorname{End}_R E)^n, E) \cong \operatorname{Hom}_{\operatorname{End}_R E}(E \oplus F, E) \\ &\cong \operatorname{Hom}_{\operatorname{End}_R E}(F, E) \oplus \operatorname{End}_{\operatorname{End}_R E}(E). \end{aligned}$$

If we define the operation of $\operatorname{End}_R E$ to be composition of mappings on the left, these become isomorphisms over R. Since E is balanced, $\operatorname{End}_{\operatorname{End}_R E}(E) \cong R$, so E is a generator.

Problem X.9(a). *Let A be an artinian commutative ring. Prove all prime ideals are maximal. [Hint: Given a prime ideal $\mathfrak{p}$, let $x \in A$, $x \notin \mathfrak{p}$. Consider the descending chain $(x) \supset (x^2) \supset \dots$.]*

Solution. We show that any artinian domain is a field. Let $\mathfrak{p}$ be a prime ideal, so that $A/\mathfrak{p}$ is a domain. Any quotient ring of an artinian ring is artinian (Proposition 7.1), so $A/\mathfrak{p}$ is artinian. Let $x \in A/\mathfrak{p}$ be nonzero. Then the descending chain $(x) \supset (x^2) \supset \dots$ must terminate, so $(x^k) = (x^{k+1})$ for some integer k; therefore there exists a $y \in A/\mathfrak{p}$ such that $x^{k+1}y = x^k$, which is to say $x^k(1 - xy) = 0$, so $xy = 1$, and $x \in (A/\mathfrak{p})^*$. Therefore $A/\mathfrak{p}$ is a field, so $\mathfrak{p}$ is maximal.

Problem X.9(b). *There is only a finite number of prime, or maximal, ideals. [Hint: Among all finite intersections of maximal ideals, pick a minimal one.]*

Solution. Let S be the set of finite intersections of maximal ideals in A. This set is nonempty, so by Exercise XVII.2(c), there exists a minimal such intersection $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r$. If $\mathfrak{m}$ is any maximal ideal of A, then, $\mathfrak{m} \cap \bigcap_i \mathfrak{m}_i = \bigcap_i \mathfrak{m}_i$ so $\mathfrak{m} \supset \bigcap_i \mathfrak{m}_i \supset \mathfrak{m}_1\mathfrak{m}_2 \dots \mathfrak{m}_r$. A maximal ideal is prime, so $\mathfrak{m} \supset \mathfrak{m}_i$ for some i, but since $\mathfrak{m}_i$ is maximal so $\mathfrak{m} = \mathfrak{m}_i$.

Date: April 1, 2003.

XVII: 12 (first sentence); X: 9, 10, 11.

Problem X.9(c). *The ideal N of nilpotent elements in A is nilpotent, that is there exists a positive integer k such that $N^k = (0)$. [Hint: Let k be such that $N^k = N^{k+1}$. Let $\mathfrak{a} = N^k$. Let $\mathfrak{b}$ be a minimal ideal such that $\mathfrak{b}\mathfrak{a} \neq 0$. Then $\mathfrak{b}$ is principal and $\mathfrak{b}\mathfrak{a} = \mathfrak{b}$.]*

Solution. Let k be such that $N^k = N^{k+1}$. Suppose that $N^k \neq 0$; let S be the set of ideals $\mathfrak{b}$ of A such that $\mathfrak{b}N^k \neq 0$. The set S is nonempty because $N^k \in S$, as $N^k N^k = N^{2k} = N^k \neq 0$. Since A is artinian, S has a minimal element $\mathfrak{b}$. There is an element $b \in \mathfrak{b}$ such that $bN^k \neq 0$; therefore $(b) \in S$ and $(b) \subset \mathfrak{b}$ so by minimality $(b) = \mathfrak{b}$, in particular, $\mathfrak{b}$ is finitely generated. But $\mathfrak{b}N^k \subset \mathfrak{b}$ and $(\mathfrak{b}N^k)N^k = \mathfrak{b}N^{2k} = \mathfrak{b}N^k$, so again by minimality, $\mathfrak{b}N^k = \mathfrak{b}$. But every element of N^k is nilpotent hence contained in every maximal ideal, so by Nakayama's lemma, $\mathfrak{b} = 0$, a contradiction.

Problem X.9(d). *A is noetherian.*

Solution. Let k be an integer such that $N^k = 0$ as in part (c). Then $\bigcap_{\mathfrak{p}} \mathfrak{p} = \sqrt{0} = N$, but by part (a) this implies $N = \bigcap_i \mathfrak{m}_i$. Let k be an integer such that $N^k = 0$. Then

$$N^k = 0 = (\textstyle\bigcap_i \mathfrak{m}_i)^k \supset (\mathfrak{m}_1 \dots \mathfrak{m}_r)^k = \mathfrak{m}_1^k \dots \mathfrak{m}_r^k.$$

Consider A as a module over itself; A is noetherian as an A-module if and only if A is noetherian as a ring. We have a filtration

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1^2 \supset \dots \supset \mathfrak{m}_1^k \dots \mathfrak{m}_r^k = 0$$

of A. At the step $E \supset E\mathfrak{m}_i$ in the filtration, $E/E\mathfrak{m}_i$ is a vector space over the field $A/\mathfrak{m}_i$ which is finite-dimensional as A is artinian (as in Exercise XVII.2(a)). Therefore A has a finite simple filtration, so by Proposition 7.2, A is noetherian as well as artinian.

Problem X.9(e). *There exists an integer r such that*

$$A \cong \prod_{\mathfrak{m}} A/\mathfrak{m}^r$$

where the product is taken over all maximal ideals.

Solution. Let r be such that $N^r = 0$, and let $\mathfrak{m}_i$ be the maximal ideals of A. Since $\mathfrak{m}_i + \mathfrak{m}_j = A$ for $i \neq j$, we also have $\mathfrak{m}_i^r + \mathfrak{m}_j^r = A$ for $i \neq j$ (otherwise, $\mathfrak{m}_i^r + \mathfrak{m}_j^r \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$; then $\mathfrak{m}_i^r \subset \mathfrak{m}$ so $\mathfrak{m}_i \subset \mathfrak{m}$, and similarly $\mathfrak{m}_j \subset \mathfrak{m}$, a contradiction). By the Chinese remainder theorem, then,

$$A \to \prod_{\mathfrak{m}} A/\mathfrak{m}_i^r$$

is surjective. It is also injective, since $\bigcap_i \mathfrak{m}_i^r = N^r = 0$ (as in part (d)), therefore it is an isomorphism.

Problem X.9(f). *We have*

$$A \cong \prod_{\mathfrak{p}} A_{\mathfrak{p}}$$

where again the product is taken over all prime ideals $\mathfrak{p}$.

Solution. It is enough to show that this map is an isomorphism considered as a map of A-modules. Let $\mathfrak{p}_i$ be the primes (maximal ideals) of A. Since localization preserves exact sequences (it is flat), it is enough to show that the map $A \to \prod_i A_{\mathfrak{p}_i}$ is an isomorphism after localization at every prime ideal $\mathfrak{p}$ of A. But in this circumstance we have the map

$$A_{\mathfrak{p}} \to \prod_i (A_{\mathfrak{p}_i})_{\mathfrak{p}} = \prod_i (A_{\mathfrak{p}})_{\mathfrak{p}_i}.$$

Now $A_{\mathfrak{p}}$ is artinian (descending chains of ideals of $A_{\mathfrak{p}}$ are descending chains of ideals of A contained in $\mathfrak{p}$) and a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, so $N = \mathfrak{p}A_{\mathfrak{p}}$ and $(\mathfrak{p}A_{\mathfrak{p}})^r = 0$. Then for $\mathfrak{p} \neq \mathfrak{p}_i$, if $x_i \in \mathfrak{p} \setminus \mathfrak{p}_i$, $x_i^r = 0$, so $(A_{\mathfrak{p}})_{\mathfrak{p}_i} = 0$ and the map is an isomorphism.

Problem X.10. *Let A, B be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$, respectively. Let $f : A \to B$ be a homomorphism. Suppose that f is* local, *i.e. $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Assume that A, B are noetherian, and assume that:*

(1) *$A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism;*
(2) *$\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective;*
(3) *B is a finite A-module, via f.*

Prove that f is surjective.

Solution. First, $\mathfrak{m}_B$ is a finitely generated B-module (since B is noetherian) and $f(\mathfrak{m}_A)$ is a finitely generated B-submodule of $\mathfrak{m}_B$ with $\mathfrak{m}_B = f(\mathfrak{m}_A) + \mathfrak{m}_B^2$ by (2). By Nakayama's lemma (X.4.2), $f(\mathfrak{m}_A) = \mathfrak{m}_B$.

Second, since B is finite over A and $f(A)$ is a B-submodule with $B = f(A) + \mathfrak{m}_B B$ by (1), since $A \to B/\mathfrak{m}_B$ is surjective. But $\mathfrak{m}_B B = \mathfrak{m}_A B$ treating B as an A-module by f, so by Nakayama's lemma, $f(A) = B$, so f is surjective.

Problem X.11. *Let A be a commutative ring and M an A-module. Define the* support *of M by*

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec} A : M_{\mathfrak{p}} \neq 0\}.$$

If M is finite over A, show that $\operatorname{Supp} M = V(\operatorname{Ann}(M))$, where $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supset \mathfrak{a}\}$ and $\operatorname{Ann}(M) = \{a \in A : aM = 0\}$.

Solution. Let $\mathfrak{p} \in \operatorname{Spec} A$ be so that $M_{\mathfrak{p}} \neq 0$. If $aM = 0$ then $aM_{\mathfrak{p}} = 0$ so if $a \notin \mathfrak{p}$ then $M_{\mathfrak{p}} = 0$; hence $\operatorname{Ann}(M) \subset \mathfrak{p}$.

Conversely, suppose $\operatorname{Ann}(M) \subset \mathfrak{p}$. Let $m_1, \dots, m_r$ generate M over A. Suppose that $M_{\mathfrak{p}} = 0$; then for all i there exists an $a_i \notin \mathfrak{p}$ such that $a_i m_i = 0 \in M$. Then $a = \prod_i a_i$ has $aM = 0$; therefore $a \in \mathfrak{p}$, a contradiction.