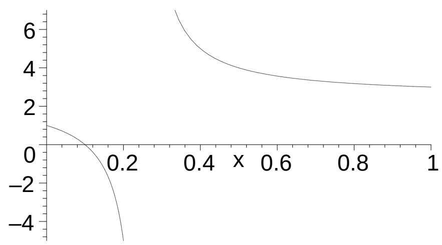
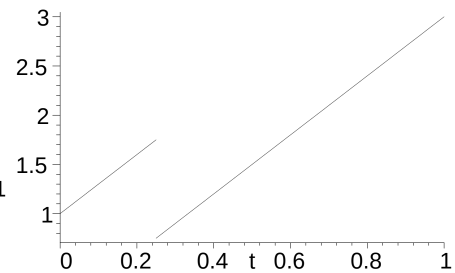
MATH 1A: CALCULUS
HOMEWORK #3

JOHN VOIGHT

§2.5: Continuity

Problem 44. These two examples are given by the following graphs:





Notice that for the graph on the left, no intermediate value (other than the endpoints) between 1 and 3 is assumed by the function in the given range. Contrarywise, the piecewise defined function on the right satisfies the conclusion but not the hypothesis of the Intermediate Value Theorem.

Problem 46. The function $f(x) = x^2$ has $f(1) = 1$ and $f(2) = 4$; it is continuous on the closed interval $[1, 2]$ (it is a polynomial). Since $1 < 2 < 4$, by the Intermediate Value Theorem, there exists a c in $(1, 2)$ such that $f(c) = c^2 = 2$.

Problem 48. Consider the function $f(x) = \sqrt[3]{x} + x - 1$. It has $f(0) = -1$ and $f(1) = \sqrt[3]{1} + 1 - 1 = 1$. It is continuous on the interval $[0, 1]$ (it contains only roots and powers), so since $-1 < 0 < 1$, by the Intermediate Value Theorem, there exists a c in $(0, 1)$ such that $\sqrt[3]{c} + c - 1 = 0$, i.e. $\sqrt[3]{c} = 1 - c$.

Problem 52(a). Consider the function $f(x) = x^5 - x^2 + 2x + 3$. We have $f(-1) = -1 - 1 - 2 + 3 = -1$ and $f(0) = 3$. It is continuous on the interval $[-1, 0]$ so by the IVT there exists a c in $(-1, 0)$ such that $f(c) = 0$, i.e. there exists a real root.

Problem 54(a). Consider the function $f(x) = \sqrt{x-5} - 1/(x + 3)$. We have $f(5) = -1/8$ and $f(6) = 8/9$. It is continuous on the closed interval $[5, 6]$ (but is not even defined for $x < 5$), so by the IVT f has a real root, hence so does our original equation.

§2.6: Limits at Infinity; Horizontal Asymptotes

Problem 4. For (a) and (b), the graph tends to the value 2 as $x \to \infty$ and -2 as $x \to -\infty$, so

$$\lim_{x\to\infty} g(x) = 2, \qquad \lim_{x\to-\infty} g(x) = -2.$$

For (c) and (d), the graph tends tends both towards the left and right of $x = 3$ to the value ∞, so $\lim_{x\to 3} g(x) = \infty$, whereas for $x = 0$ it tends on both sides to the value $-\infty$, so $\lim_{x\to 0} g(x) = -\infty$.

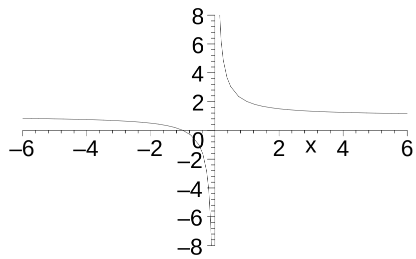
Date: February 10, 2004.

§2.5: 44, 46, 48, 52(a), 54(a); §2.6: 4, 6, 12, 16, 18, 26, 52, 60; §2.7: 2, 6(a)(i), 6(b), 10, 12; §2.8: 4, 6, 8, 10(a), 18, 22.

For (e), for x to the right of -2 the graph tends downward, so $\lim_{x\to -2^+} g(x) = -\infty$. (Note that $\lim_{x\to -2^-} g(x) = 1$, so $\lim_{x\to 2} g(x)$ is undefined!)

For (f), we note that the vertical asymptotes occur at $x = -2, x = 0, x = 3$, and the horizontal asymptotes at $y = -2, y = 2$.

Problem 6. Directly translating the conditions, we see that such a function can be given as:



Problem 12. Following Example 3, we must first divide the numerator and denominator by the highest power of x that occurs in the denominator of our rational function, i.e. x^3:

$$\lim_{x\to\infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}} = \lim_{x\to\infty} \sqrt{\frac{\dfrac{12x^3 - 5x + 2}{x^3}}{\dfrac{3x^3 + 4x^2 + 1}{x^3}}} = \lim_{x\to\infty} \sqrt{\frac{12 - \dfrac{5}{x^2} + \dfrac{2}{x^3}}{3 + \dfrac{4}{x} + \dfrac{1}{x^3}}}.$$

By Limit Law 11 (roots), we have that this is equal to

$$\sqrt{\lim_{x\to\infty} \frac{12 - \dfrac{5}{x^2} + \dfrac{2}{x^3}}{3 + \dfrac{4}{x} + \dfrac{1}{x^3}}}.$$

By Limit Law 5 (ratio), since the denominator does not tend to 0 as $x \to \infty$, we obtain

$$\sqrt{\frac{\lim\limits_{x\to\infty} \left(12 - \dfrac{5}{x^2} + \dfrac{2}{x^3}\right)}{\lim\limits_{x\to\infty} \left(3 + \dfrac{4}{x} + \dfrac{1}{x^3}\right)}}.$$

By Limit Laws 1,2,3 (addition, subtraction, constant), we may take the limit of each factor separately, and obtain

$$\sqrt{\frac{12 - 0 + 0}{3 + 0 + 0}} = \sqrt{4} = 2.$$

Problem 16. Dividing top and bottom by y^2 we obtain

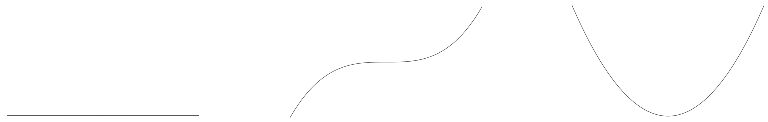
$$\lim_{y\to\infty} \frac{2 - 3y^2}{5y^2 + 4y} = \lim_{y\to\infty} \frac{2/y^2 - 3}{5 + 4/y} = -\frac{3}{5}.$$

Problem 18. Dividing top and bottom by t^3 we get

$$\lim_{t\to-\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} = \lim_{t\to-\infty} \frac{1/t + 2/t^3}{1 + 1/t - 1/t^3} = 0.$$

Problem 26. The value of $\cos x$ oscillates infinitely often as $x \to \infty$ between -1 and 1, therefore the limit does not exist.

Problem 52. For (i), we have $x^0 = 1$, and this is a constant function (a line). For (ii), $n > 0$, n odd, we have a graph that looks like $y = x^3$, for example; for (iii), $n > 0$, n even, we have a graph that looks like $y = x^2$.



(i) $n = 0$ (ii) $n > 0$, n odd $n > 0$, n even

Thus for $n = 0$, we have

$$\lim_{x\to 0^+} x^n = \lim_{x\to 0^-} x^n = \lim_{x\to\infty} x^n = \lim_{x\to -\infty} x^n = 1;$$

for (ii) we have

$$\lim_{x\to 0^+} x^n = \lim_{x\to 0^-} x^n = 0, \qquad \lim_{x\to\infty} x^n = \infty, \qquad \lim_{x\to -\infty} x^n = -\infty;$$

and for (iii) we have

$$\lim_{x\to 0^+} x^n = \lim_{x\to 0^-} x^n = 0, \qquad \lim_{x\to\infty} x^n = \lim_{x\to -\infty} x^n = \infty.$$

We obtain the following graphs for (iv) and (v):

(iv) $n < 0$, n odd (v) $n < 0$, n even

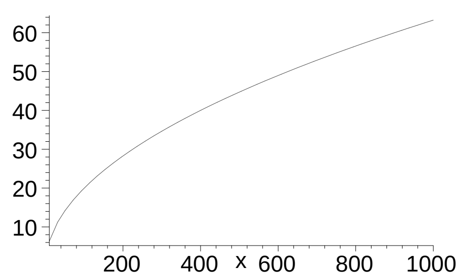
Therefore for (iv) we have

$$\lim_{x\to 0^+} x^n = \infty, \qquad \lim_{x\to 0^-} x^n = -\infty, \qquad \lim_{x\to\infty} x^n = \lim_{x\to -\infty} x^n = 0;$$

for (v) we have

$$\lim_{x\to 0^+} x^n = \lim_{x\to 0^-} x^n = \infty, \qquad \lim_{x\to\infty} x^n = \lim_{x\to -\infty} x^n = 0.$$

Problem 60. We need to find a value N such that $f(x) > M = 100$ whenever $x > N$. Sketching f by plotting points, we obtain the following.



Notice that f is increasing, and that f grows like a square root function e.g. $f(x) = \sqrt{x}$. Therefore if we want to find a value such that $f(x) > M$, it is worth trying the values $N = M^2 = 10000$. Indeed, we find that

$$\frac{2(10000) + 1}{\sqrt{10001}} \approx 200$$

which works.

More precisely, we claim that

$$f(x) = \frac{2x+1}{\sqrt{x+1}} > \sqrt{2x}.$$

To prove this, we multiply out: the foregoing inequality is true if and only if

$$2x + 1 > \sqrt{2x(x+1)}$$

which is the same after squaring as

$$(2x+1)^2 = 4x^2 + 4x + 1 > 2x^2 + 2x$$

which is true for $x > 1$. Therefore

$$f(x) > \sqrt{2x}$$

so we can take $x > N = 5000$, for then $f(x) > \sqrt{2(5000)} = 100$.

§2.7: Tangents, Velocities, and Other Rates of Change

Problem 2. The average velocity is given by the ratio of distance travelled over time:

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

The instantaneous velocity is the limit of the average velocities over shrinking time intervals:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Problem 6(a)(i). According to Definition 1, the slope is

$$m = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{x^3 + 1}{x + 1} = \lim_{x \to -1} x^2 - x + 1 = 3.$$

Problem 6(b). By the point-slope formula, the tangent line is $y + 1 = 3(x + 1)$, or $y = 3x + 2$.

Problem 10. We compute the slope:

$$m = \lim_{x \to 0} \frac{2x/(x+1)^2 - 0}{x - 0} = \lim_{x \to 0} \frac{2}{(x+1)^2} = 2.$$

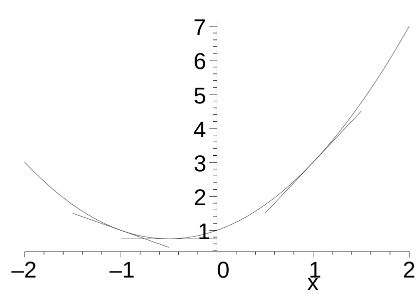
By the point-slope formula, the tangent line is $y = 2x$.

Problem 12. For (a), the slope is given by

$$\begin{aligned}\lim_{x \to a} \frac{(1 + x + x^2) - (1 + a + a^2)}{x - a} &= \lim_{x \to a} \frac{x^2 - a^2 + x - a}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a+1)}{x - a} \\ &= \lim_{x \to a} x + a + 1 = 2a + 1.\end{aligned}$$

For (b), we find that this slope is: (i) -1; (ii) 0; (iii) 3.

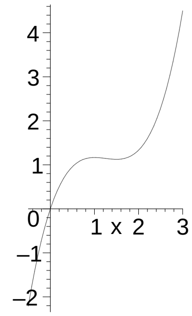
For (c), we have the graph:



§2.8: Derivatives

Problem 4. We have $y = f(4) = 3$.

If the tangent line goes through $(4, 3)$ and $(0, 2)$, then it has slope $(2 - 3)/(0 - 4) = 1/4 = f'(4)$.



Problem 6. The requirement that $g(0) = 0$ says that the graph should go through the origins; the condition that $g'(0) = 3$ says that the slope of the graph at the point $(0, 0)$ is 3, and similarly for the last two conditions.

Problem 8. We have

$$g'(0) = \lim_{h\to 0} \frac{(1 - h^3) - 1}{h} = \lim_{h\to 0}(-h^2) = 0.$$

So by the point-slope formula, this means the tangent line is given by $y = 1$.

Problem 10(a). We have

$$\begin{aligned}
G'(a) &= \lim_{h\to 0} \frac{(a + h)/(1 + 2(a + h)) - a/(1 + 2a)}{h} \\
&= \lim_{h\to 0} \frac{(a + h)(2a + 1) - a(2a + 2h + 1)}{h(2a + 2h + 1)(2a + 1)} \\
&= \lim_{h\to 0} \frac{a + h + 2a^2 + 2ah - 2a^2 - 2ah - a}{h(2a + 2h + 1)(2a + 1)} \\
&= \lim_{h\to 0} \frac{h}{h(2a + 2h + 1)(2a + 1)} = \lim_{h\to 0} \frac{1}{(2a + 2h + 1)(2a + 1)} = \frac{1}{(2a + 1)^2}.
\end{aligned}$$

Therefore the slope of the tangent line at the point $(-1/4, -1/2)$ is $G'(-1/4) = 1/(-1/2 + 1)^2 = 4$, and the equation is $y + 1/2 = 4(x + 1/4)$ or just $y = 4x + 1/2$.

Problem 18. Rationalizing the numerator, we obtain:

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{3(a + h) + 1} - \sqrt{3a + 1}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{3a + 3h + 1} - \sqrt{3a + 1})(\sqrt{3a + 3h + 1} + \sqrt{3a + 1})}{h(\sqrt{3a + 3h + 1} + \sqrt{3a + 1})} \\
&= \lim_{h\to 0} \frac{(3a + 3h + 1) - (3a + 1)}{h(\sqrt{3a + 3h + 1} + \sqrt{3a + 1})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3a + 3h + 1} + \sqrt{3a + 1}} = \frac{3}{2\sqrt{3a + 1}}.
\end{aligned}$$

Problem 22. From the template

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

we have $f(x) = \tan x$ and $a = \pi/4$. Indeed, $\tan(\pi/4) = 1$.