MATH 1A: CALCULUS
HOMEWORK #8

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§4.2: The Mean Value Theorem

Problem 2. We compute that $f(0) = 5 = 2^3 - 3(2^2) + 2(2) + 5 = f(2)$, so the starting and ending values are the same. The function f is continuous on $[0, 2]$ (in fact, on all real numbers) because it is a polynomial. It is differentiable on $(0, 2)$ also because it is a polynomial: to check this, we see that

$$f'(x) = 3x^2 - 6x + 2$$

which has domain all real numbers.

We want to find all values c such that $f'(c) = 0$:

$$f'(c) = 3c^3 - 6c + 2 = 0;$$

by the quadratic formula, we get

$$c = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{\sqrt{3}}{3}.$$

In terms of Rolle's theorem, for the conclusion we consider only values in the open interval $(0, 2)$, and both of these values lie in this interval (since $\sqrt{3}/3 = 1/\sqrt{3} < 1$).

Problem 4. We see that

$$f(-6) = 0 = f(0);$$

the function f is continuous on the interval $[-6, 0]$ since it is the product of a root function and a polynomial; and that

$$f'(x) = (x(x+6)^{1/2})' = \frac{1}{2}x(x+6)^{-1/2} + (x+6)^{1/2} = \frac{x}{2\sqrt{x+6}} + \sqrt{x+6}$$

which has domain $x > -6$ so f is differentiable on $(-6, 0)$. Therefore f satisfies the conditions of Rolle's theorem, and

$$f'(x) = \frac{x + 2(x+6)}{2\sqrt{x+6}} = 0$$
$$3x + 12 = 0$$

so $c = -4$, which is indeed in $(-6, 0)$.

Problem 6. We compute that $f'(x) = -2(x-1)^{-3} = -2/(x-1)^3$, which is never zero. This does not contradict Rolle's theorem because the function f is discontinuous at $x = 1$, so Rolle's theorem does not apply to the function f on the interval $[0, 2]$.

Problem 8. We see that $f(1) = 5$ and $f(7) = 2$, so we look for values c such that

$$f'(c) = \frac{f(7) - f(1)}{7 - 1} = \frac{2 - 5}{6} = -\frac{1}{2}.$$

Looking at the graph, we see that the slope is about $-1/2$ and we have the values

$$c \approx 1.2, 2.8, 4.7, 5.8.$$

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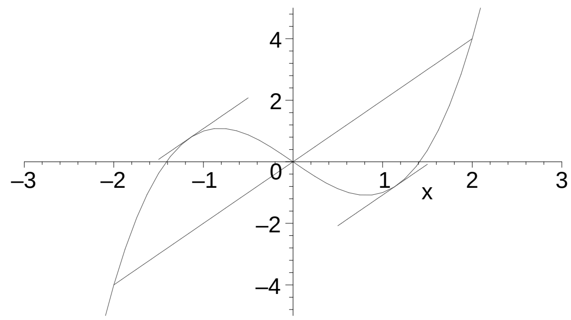
§4.2: 2, 4, 6, 8, 10, 12, 14, 16, 18, 19, 21(a), 24, 30, 31, 32; §4.3: 2, 6, 8, 12, 14, 20, 22, 32, 34, 38, 42; Updated March 17, 2004.

Problem 10(a). By plotting points, we have:



From the graph, we estimate that the x coordinates are $-1.1, 1.1$.

Problem 10(b). On the interval $[-2, 2]$, we compute

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - (-4)}{2 - (-2)} = 2.$$

So we want to solve

$$f'(x) = 3x^2 - 2 = 2$$

which has the roots $x^2 = 4/3$ or $x = \pm 2/\sqrt{3} = \pm(2\sqrt{3})/3 \approx \pm 1.15$. These compare well with the value estimated in (a).

Problem 12. The function $f(x) = x^3 + x - 1$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ because it is a polynomial, therefore it satisfies the hypotheses of the Mean Value Theorem.

We compute that

$$\frac{f(2) - f(0)}{2 - 0} = \frac{9 + 1}{2} = 5$$

and

$$\begin{aligned} f'(x) = 3x^2 + 1 &= 5 \\ x^2 &= 4/3 \\ x &= \pm 2/\sqrt{3} = \pm(2\sqrt{3})/3. \end{aligned}$$

In the interval $(0, 2)$, we have only $c = (2\sqrt{3})/3$ satisfying the conditions of the theorem.

Problem 14. The function $f(x) = x/(x + 2)$ is continuous on $[1, 4]$ because this is a rational function on its domain $(x \neq -2)$. It is differentiable on $(1, 4)$ because

$$f'(x) = \frac{(x + 2) - x}{(x + 2)^2} = \frac{2}{(x + 2)^2}$$

is also a rational function with domain $x \neq -2$. Therefore it satisfies the conditions of the Mean Value Theorem.

We compute that

$$\frac{f(4) - f(1)}{4 - 1} = \frac{2/3 - 1/3}{3} = \frac{1}{9}$$

and

$$\begin{aligned} f'(x) = \frac{2}{(x+2)^2} &= \frac{1}{9} \\ 18 &= (x+2)^2 \\ \pm 3\sqrt{2} &= x+2 \\ x &= -2 \pm 3\sqrt{2} \end{aligned}$$

In the interval $(1,4)$, we take only $c = -2 + 3\sqrt{2} \approx 2.2$ satisfying the conditions of the theorem.

Problem 16. We see that

$$f(2) - f(0) = 3 + 1 = 4$$

and

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

so we look for c such that $4 = f'(c)(2)$ or $f'(c) = 2$, i.e.

$$\begin{aligned} \frac{-2}{(c-1)^2} &= 2 \\ -1 &= (c-1)^2 \end{aligned}$$

This equation has no solution, since the square of a real number is never negative. Therefore no such c satisfying the Mean Value Theorem exists.

This does not contradict the Mean Value Theorem because the function f is discontinuous at $x = 1$, so the Mean Value Theorem does not apply to the function f on the interval $[0,2]$.

Problem 18. First, we show that $2x - 1 - \sin x = 0$ has at least one real root. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1$ and $f(\pi) = 2\pi - 1 > 0$. The function f is continuous on the interval $[0,\pi]$ (in fact, all real numbers) because it is the difference of a polynomial and a trigonometric function. Therefore by the Intermediate Value Theorem, it has a root.

Suppose the equation has two real roots. The function f is continuous and differentiable for all real numbers (since $f'(x) = 2 - \cos x$ arises from a trigonometric function). Therefore if $f(a) = f(b) = 0$, f satisfies the conditions of Rolle's theorem on the interval $[a,b]$; but

$$f'(x) = 2 - \cos x = 0$$

has no solution, since $\cos x \leq 1$. This is a contradiction, so f has at most one real root. Hence f has exactly one real root.

Problem 19. Suppose that f has more than one real root in the interval $[-2,2]$. The function f is continuous and differentiable on this interval (it is a polynomial), so by Rolle's theorem, $f'(x) = 0$ somewhere in this interval. But

$$f'(x) = 3x^2 - 15 = 3(x^2 - 5) = 0$$

has only the roots $x = \pm\sqrt{5} \approx 2.23$, which do not lie in the interval $(-2,2)$. This is a contradiction, so f has at most one real root.

Problem 21(a). Suppose that the cubic polynomial f has at least 4 roots. A polynomial is continuous and differentiable for all real numbers, therefore by Rolle's theorem, the derivative $f'(x)$ must take on the value 0 at least 3 times, in each of the three consecutive intervals with endpoints among the 4 roots. But since f has degree 3, we know that f' has degree 2, which by the quadratic formula can have at most 2 roots. This is a contradiction, therefore f has at most 3 real roots.

Problem 24. If we can apply the Mean Value Theorem (which we must be able to, see Example 5), we will conclude that there exists a c in (a,b) such that

$$f'(c)(b-a) = f(b) - f(a).$$

This looks like the inequality

$$18 \le f(8) - f(2) \le 30$$

if we take $b = 8$ and $a = 2$.

Therefore, let $a = 2$ and $b = 8$, i.e. look at f on the interval $[2, 8]$. By the Mean Value Theorem, there is a c in $(2, 8)$ such that

$$6f'(c) = f(8) - f(2)$$

But now we are given that

$$3 \le f'(c) \le 5,$$

which multiplying by 6 gives

$$18 \le 6f'(c) \le 30$$

which by the previous equality is just

$$18 \le f(8) - f(2) \le 30$$

as desired.

Problem 30. Let $g(x) = cx$. Then $g'(x) = c$, so $f'(x) = g'(x)$ for all real numbers. Therefore by Corollary 7, $f - g$ is a constant, which we call d. That is,

$$f(x) = g(x) + d = cx + d.$$

Problem 31. We see indeed that $f'(x) = -1/x^2$ and that in both cases $x > 0$ and $x < 0$, $g'(x) = -1/x^2$.

We can conclude from Corollary 7 that $f - g$ is constant on the interval $(0, \infty)$, and separately that $f - g$ is also constant on the interval $(-\infty, 0)$, but these constants may very well be different, since neither f nor g is defined for $x = 0$.

Problem 32. Let

$$f(x) = 2\sin^{-1} x$$

and

$$g(x) = \cos^{-1}(1 - 2x^2).$$

for $x \ge 0$. By trig formulas, we have

$$f'(x) = \frac{2}{\sqrt{1 - x^2}}.$$

By the chain rule, with $u = 1 - 2x^2$ we compute that

$$\begin{aligned} g'(x) &= -\frac{1}{\sqrt{1-u^2}}u' = -\frac{1}{\sqrt{1-(1-2x^2)^2}}(-4x) = \frac{4x}{\sqrt{1-(1-4x^2+4x^4)}} \\ &= \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2(1-x^2)}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}. \end{aligned}$$

Since $f'(x) = g'(x)$, i.e. $f'(x) - g'(x) = 0$, by Corollary 7, we see that $f(x) - g(x)$ is constant, or $f(x) = g(x) + c$. Now

$$f(0) = 2\sin^{-1}(0) = 0$$

and

$$g(0) = \cos^{-1}(1) = 0 = 0$$

so in fact $c = 0$, i.e.

$$2\sin^{-1} x = \cos^{-1}(1 - 2x^2).$$

§4.3: How derivatives affect the shape of a graph

Problem 2(a). f is increasing on $(1, 3.9)$ and $(5, 6.5)$, approximately. (Note they ask for *open* intervals!)

Problem 2(b). f is decreasing on $(0, 1)$, $(3.9, 5)$, $(6.5, 9)$, approximately.

Problem 2(c). f is concave upward on $(0, 3)$ and $(8, 9)$.

Problem 2(d). f is concave downward on $(3, 8)$.

Problem 2(e). f has only the inflection point $x = 3$. At $x = 5$, it stays from concave down to concave down; at $x = 8$, it is not continuous.

Problem 6(a). f is increasing if $f' > 0$ (including appropriate endpoints); therefore f is increasing on $[0, 1]$ and $[3, 5]$.

f is decreasing if $f' < 0$ (including appropriate endpoints); therefore f is decreasing on $[1, 3]$ and $[5, 6]$.

Problem 6(b). f has a local maximum or minimum only if $f'(x) = 0$, i.e. $x = 1, 3, 5$. At $x = 1$ and $x = 5$, f' goes from $+$ to $-$ so it is a local maximum; at $x = 3$ it goes from $-$ to $+$ so it is a local minimum.

Problem 8(a). f is increasing if $f' > 0$, including appropriate endpoints; therefore f is increasing on $[2, 4]$ and $[6, 9]$. (By default, if there is no dot at $x = 0$ or $x = 9$, we assume it is defined there, i.e. it is a solid dot.)

Problem 8(b). At $x = 2$ and $x = 6$, f' goes from $-$ to $+$, so f has a local minimum there; at $x = 4$, f' goes from $+$ to $-$, so f has a local maximum there.

Problem 8(c). f is concave upward if $f'' > 0$ and concave downward if $f'' < 0$. Looking at the graph of f', we see that the slope of f' is positive on $(1, 3)$, $(5, 7)$, and $(8, 9)$ so the function is concave upward on these intervals, and f' has negative slope on $(0, 1)$, $(3, 5)$, and $(7, 8)$, so f is concave downward there.

Problem 8(d). The inflection points are where $f''(x) = 0$ and changes sign; these are the points $x = 1, 3, 5, 7, 8$.

Problem 12(a). We compute that

$$f'(x) = -6x + 3x^2 = 3x(x - 2).$$

Therefore $f'(x) > 0$ and f is increasing when $3x > 0$ and $x - 2 > 0$, i.e. $x > 2$, or when $3x < 0$ and $x - 2 < 0$, i.e. $x < 0$. Similarly, $f'(x) < 0$ and f is decreasing when $3x > 0$ and $x - 2 < 0$, i.e. $0 < x < 2$, or $3x < 0$ and $x - 2 > 0$, which cannot happen.

f is increasing on the intervals $(-\infty, 0]$ and $[2, \infty)$ and f is decreasing on the interval $[0, 2]$.

Problem 12(b). We see that $f'(x) = 0$ for $x = 0, 2$. Here,

$$f''(x) = -6 + 6x = 6x - 6$$

so $f''(0) = -6 < 0$ and $x = 0$ is a local maximum, and $f''(2) = 6 > 0$ so $x = 2$ is a local minimum.

Problem 12(c). We have $f''(x) = 6x - 6 = 0$ for $x = 1$. Since from (b) $f''(0) < 0$ and $f''(2) > 0$, we see that on the interval $(-\infty, 1)$, f is concave downward, and on the interval $(1, \infty)$, f is concave upward. f has an inflection point at $x = 1$.

Problem 14(a). We compute that

$$f'(x) = \frac{(x^2 + 3)(2x) - x^2(2x)}{(x^2 + 3)^2} = \frac{6x}{(x^2 + 3)^2}.$$

Since $(x^2 + 3)^2 > 0$ for all x, we see that $f'(x) > 0$ and f is increasing for $x > 0$ and $f'(x) < 0$ and f is decreasing for $x < 0$. Therefore f is decreasing on the interval $(-\infty, 0]$ and is increasing on the interval $[0, \infty)$.

Problem 14(b). Since f changes from decreasing to increasing at $x = 0$, we see that it is a local minimum.

Problem 14(c). We compute that

$$\begin{aligned} f''(x) &= \frac{(x^2+3)^2(6) - (6x)(2(x^2+3)(2x))}{(x^2+3)^4} \\ &= \frac{6(x^2+3)^2 - 24x^2(x^2+3)}{(x^2+3)^4} \\ &= \frac{(x^2+3)(6(x^2+3) - 24x^2)}{(x^2+3)^4} \\ &= \frac{-18x^2+18}{(x^2+3)^3} = \frac{-18(x^2-1)}{(x^2+3)^3}. \end{aligned}$$

Therefore $f''(x) = 0$ for $x = -1, 1$. We saw that $f''(0) > 0$, so f is concave upward on that interval. On the interval $(-\infty, -1)$ we see e.g. $f''(-2) = -18(3)/7^3 < 0$ so f is concave downward, and similarly $f''(2) > 0$ so f is also concave downward there.

In sum, f is concave upward on $(-1, 1)$ and f is concave downward on $(-\infty, -1)$ and $(1, \infty)$. f has inflection points at $x = \pm 1$.

Problem 20(a). The domain of the function, since $\ln x$ is defined only for $x > 0$, is $x > 0$. We have

$$f'(x) = x(1/x) + \ln x = 1 + \ln x$$

so $f'(x) < 0$ and f is decreasing for $1 + \ln x < 0$, i.e. $\ln x < -1$ or $x < e^{-1} = 1/e$, and $f'(x) > 0$ and f is increasing for $x > 1/e$.

In sum, f is decreasing on $(0, 1/e]$ and is increasing on $[1/e, \infty)$.

Problem 20(b). Now

$$f''(x) = 1/x$$

so $f''(1/e) = e > 0$, so $x = 1/e$ is a local minimum.

Problem 20(c). Since $f''(x) = 1/x$ is never zero, and f is defined only for $x > 0$, we conclude that f is always concave upward, and has no inflection point.

Problem 22(a). For $f(x) = x/(x^2+4)$, we have

$$f'(x) = \frac{(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}.$$

This is zero only when $-x^2 + 4 = 0$, i.e. $x = \pm 2$.

Since $f'(-3) = (-9+4)/(9+4)^2 < 0$ and $f'(-1) = (-1+4)/(1+4)^2 > 0$, by the First Derivative Test, $x = -2$ is a local minimum. Since $f'(1) = (-1+4)/(1+4)^2 > 0$ and $f'(3) = (-9+4)/(9+4)^2 < 0$, $x = 2$ is a local maximum.

For the Second Derivative Test, we compute that

$$\begin{aligned} f''(x) &= \frac{(x^2+4)^2(-2x) - (-x^2+4)(2(x^2+4)(2x))}{(x^2+4)^4} \\ &= \frac{-2x(x^2+4) - 4x(-x^2+4)(x^2+4)}{(x^2+4)^4} \end{aligned}$$

so $f''(-2) = (4(8)-0)/(4+4)^4 > 0$, and again $x = -2$ is a local minimum, and $f''(2) = (-4(8)-0)/(4+4)^4 < 0$, so $x = 2$ is a local maximum.

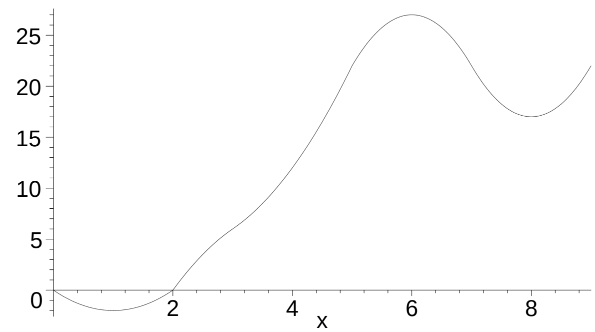
Problem 32(a). f is increasing when $f' > 0$, i.e. on $[1, 6]$ and $[8, 9]$, and decreasing when $f' < 0$, i.e. on $[0, 1]$ and $[6, 8]$.

Problem 32(b). x has a local maximum or minimum possibly only when $f' = 0$, when $x = 1, 6, 8$. We see that $f''(1) > 0$ (the slope of the graph of f' at $x = 1$ is positive), so $x = 1$ is a local minimum, and similarly $f''(6) < 0$ so $x = 6$ is a local maximum and $f''(8) > 0$ so $x = 8$ is a local minimum.

Problem 32(c). f is concave upward where $f'' > 0$, on the intervals $(0, 2)$, $(3, 5)$, and $(7, 9)$. f is concave downward where $f'' < 0$, on $(2, 3)$ and $(5, 7)$.

Problem 32(d). The points of inflection are where $f'' = 0$ and f'' changes sign. These are the values $x = 2, 3, 5, 7$.

Problem 32(e). We use the data from (a)–(d): first, we locate the local maximum at $x = 1$ and local minima at $x = 6$ and $x = 8$; then we note it is concave upward on $[0, 2)$, $(3, 5)$, and $(7, 9]$ and concave downward on $(2, 3)$ and $(5, 7)$. Since f is increasing on $[1, 6]$ and $[8, 9]$ and decreasing on $[0, 1]$ and $[6, 8]$, starting at the origin we can draw the following graph:



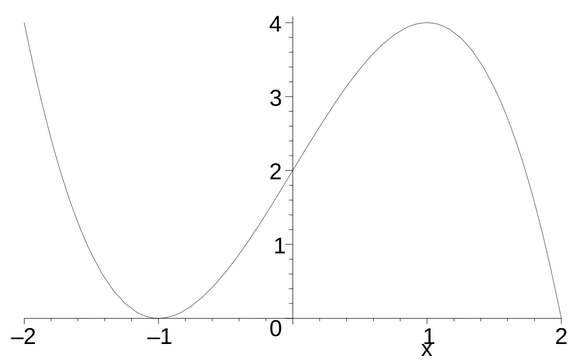
Problem 34(a). We have $f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1 - x)(1 + x)$, so $f' > 0$ and f is increasing when $1 - x > 0$ and $1 + x > 0$, i.e. when $x < 1$ and $x > -1$ or on the interval $(-1, 1)$, or when $1 - x < 0$ and $1 + x < 0$, i.e. $x > 1$ and $x < -1$ which cannot happen.

We see then that f is decreasing on $(-\infty, -1]$ and $[1, \infty)$, and increasing on $[-1, 1]$.

Problem 34(b). We have $f'(x) = 0$ for $x = -1, 1$. Since $f''(x) = -6x$, $f''(-1) = 6$ so $x = -1$ is a local minimum and $f''(1) = -6$ so $x = 1$ is a local maximum.

Problem 34(c). We have $f''(x) = -6x = 0$ for $x = 0$, the only inflection point. For $x < 0$ $f''(x) > 0$ so f is concave upward; for $x > 0$, $f''(x) < 0$ so f is concave downward.

Problem 34(d). We graph using the data in (a)–(d).



Problem 38(a). We have $h'(x) = 3(x^2 - 1)(2x) = 6x(x^2 - 1)^2$. Since $(x^2 - 1)^2 > 0$, we see that $h' > 0$ and h is increasing for $x > 0$ and $h' < 0$ and h is decreasing for $x < 0$, i.e. h is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Problem 38(b). The solutions to $h'(x) = 6x(x^2 - 1)^2 = 0$ are $x = 0, -1, 1$. We have

$$h''(x) = 6(x^2 - 1)^2 + 6x(2)(x^2 - 1)(2x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1)$$

and $h''(0) = 6 > 0$ so $x = 0$ is a local minimum. However, $h''(-1) = 0$ which is inconclusive. So we test that $h'(-1/2) = -3(1/4-1)^2 < 0$ and $h'(-3/2) = -9(9/4-1)^2 < 0$, so $x = -1$ is neither a local minimum nor a local maximum. Similary, we see that $h'(1/2) = 3(1/4-1) > 0$ and $h'(3/2) = 9(9/4-1)^2 > 0$, so $x = 1$ is neither a local minimum nor a local maximum.

Problem 38(c). We factor

$$h''(x) = (x^2-1)(6(x^2-1) - 24x^2) = (x^2-1)(30x^2-6) = 6(x^2-1)(5x^2-1) = 0.$$

This gives the roots $x = \pm 1, \pm 1/\sqrt{5}$. We test the values

$$h''(-2) = 6(4-1)(12+1) > 0$$

and

$$h''(-2/3) = 6(4/9-1)(5(4/9)-1)) < 0$$

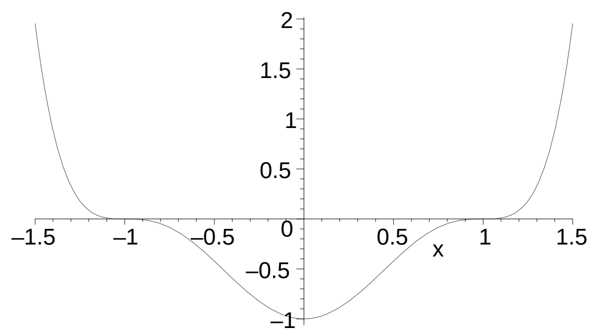
and

$$h''(0) = 6(-1)(1) > 0$$

so $x = -1, -1/\sqrt{5}$ are both inflection points. The same calculation shows that $x = 1, 1/\sqrt{5}$ are also inflection points.

The function is concave upward on $(-\infty, -1)$, $(-1/\sqrt{5}, 1/\sqrt{5})$ and $(1, \infty)$ and is concave downward on $(-1, -1/\sqrt{5})$ and $(1/\sqrt{5}, 1)$. The inflection points are at $x = \pm 1, \pm 1/\sqrt{5}$.

Problem 38(d). Using the data from (a)–(c), we have the following graph:



Problem 42(a). We have

$$f'(x) = \frac{4x^3}{x^4 + 27}$$

so since $x^4 + 27 > 0$ always, we have $f' > 0$ and f is increasing for $x > 0$ and $f' < 0$ and f is decreasing for $x < 0$.

Problem 42(b). We have $f'(x) = 0$ only for $x = 0$. We see $f'(-1) = -4/28 < 0$ and $f'(1) = 4/28 > 0$ so $x = 0$ is a local minimum.

Problem 42(c). We compute

$$\begin{aligned} f''(x) &= \frac{(x^4+27)(12x^2) - (4x^3)(4x^3)}{(x^4+27)^2} \\ &= \frac{-4x^6 + 324x^2}{(x^4+27)^2} \end{aligned}$$

This has

$$\begin{aligned} f''(x) &= -4x^6 + 324x^2 = -4x^2(x^4 - 81) = -4x^2(x^2-9)(x^2+9) \\ &= -4x^2(x-3)(x+3)(x^2+9) \end{aligned}$$

so we must consider the possible values $x = 0, 3, -3$. We see that

$$\begin{aligned} f''(-4) &= -4(16)(-7)(-1)(16+9) < 0 \\ f''(-1) &= -4(1)(-4)(2)(16+9) > 0 \\ f''(1) &= -4(1)(-2)(4)(1+9) > 0 \\ f''(4) &= -4(16)(1)(7)(16+9) < 0 \end{aligned}$$

So $x = -3, 3$ are inflection points and $x = 0$ is not. The function f is concave downward on $(-\infty, -3)$ and $(3, \infty)$ and concave upward on the interval $(-3, 3)$.

Problem 42(d). Using (a)–(d), we have: