MATH 1A: CALCULUS
HOMEWORK #13

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§5.3: The Fundamental Theorem of Calculus

Problem 4. For (a), we note that $g(-3) = 0$ (the integral is empty) and $g(3) = 0$ as well (the areas above and below the axis are equal).

For (b), we estimate that

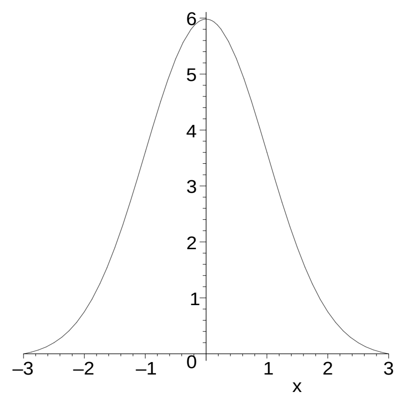
$$g(-2) = \int_{-3}^{-2} f(t)\,dt$$

is about the area of a triangle with base 1 and height 2, so $g(-2) \approx 1$. For $g(-1)$ we add approximately $2\frac{1}{2}$ (counting boxes) so $g(-1) = 1 + 2\frac{1}{2} \approx 3\frac{1}{2}$. For $g(0)$ we add finally a triangle which has area $3/2$, so $g(0) = 3\frac{1}{2} + 1\frac{1}{2} \approx 6$.

For (c), we note that g is increasing as long as we are adding more area, i.e. when the function f is positive. Therefore g is increasing on the interval $(-3, 0)$ and is decreasing on $(0, 3)$.

For (d), we note that g has a maximum value at $x = 0$ by (c) and the first derivative test.

For (e), we have the following graph, using parts (a)–(d):



For (f), note that the graph of g' and the graph of f are the same by FTC.

Problem 8. Let $f(t) = \ln t$ so $g(x) = \int_1^x f(t)\,dt$. Then by FTC we have $g'(x) = f(x) = \ln x$.

Problem 10. Let $f(x) = 1/(x+x^2)$ so $g(u) = \int_3^u f(x)\,dx$. Then by FTC we have $g'(u) = f(u) = 1/(u+u^2)$.

Problem 16. Let $f(t) = (t + \sin t)$ and $h(x) = \cos x$. Then by FTC we have

$$y' = f(h(x))h'(x) = (\cos x + \sin(\cos x))\,(-\sin x) = -\sin x(\cos x + \sin(\cos x)).$$

Problem 22.

$$\int_0^4 (1 + 3y - y^2)\,dy = \left(y + 3\frac{y^2}{2} - \frac{y^3}{3}\right)_0^4 = 4 + 3 \cdot 8 - \frac{64}{3} - 0 = \frac{20}{3}.$$

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§5.3: 4, 8, 10, 16, 22, 26, 30, 32, 40, 42, 50, 52, 62, 66; §5.4: 2, 4, 22, 28, 38, 40, 44, 48, 54, 56, 58.

Problem 26. The integral does not exist because the function $f(x) = x^{-5}$ has a vertical asymptote at $x = 0$ and hence is discontinuous on the interval $[-2, 3]$.

Problem 30.

$$\int_1^4 x^{-1/2}\,dx = 2x^{1/2}|_1^4 = 2(2-1) = 2.$$

Problem 32.

$$\int_0^1 (3 + x^{3/2})\,dx = \left(3x + \frac{2}{5}x^{5/2}\right)_0^1 = 3 + \frac{2}{5} - 0 = \frac{17}{5}.$$

Problem 40.

$$\int_1^2 (4u^{-3} + u^{-1})\,du = \left(\frac{4}{-2}u^{-2} + \ln|u|\right)_1^2 = \left(-\frac{2}{u^2} + \ln u\right)_1^2 = -\frac{1}{2} + \ln 2 + 2 - 0 = \frac{3}{2} + \ln 2.$$

Problem 42.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} x\,dx + \int_0^{\pi} \sin x\,dx = \frac{x^2}{2}\bigg|_{-\pi}^{0} - \cos x|_0^{\pi} = 0 - \frac{\pi^2}{2} - (\cos\pi - \cos 0) = -\frac{\pi^2}{2} + 2.$$

Problem 50. Let $h_1(x) = \tan x$ and $h_2(x) = x^2$, $f(t) = 1/\sqrt{2+t^4}$. Then by the FTC,

$$g'(x) = f(h_2(x))h_2'(x) - f(h_1(x))h_1'(x) = \frac{1}{\sqrt{2+x^8}}(2x) - \frac{1}{\sqrt{2+\tan^4 x}}(\sec^2 x) = \frac{2x}{\sqrt{2+x^8}} - \frac{\sec^2 x}{\sqrt{2+\tan^4 x}}.$$

Problem 52. Let $h_1(x) = \cos x$ and $h_2(x) = 5x$, $f(u) = \cos(u^2)$. Then by the FTC,
$g'(x) = f(h_2(x))h_2'(x) - f(h_1(x))h_1'(x) = \cos((5x)^2)(5) - \cos(\cos^2 x)(-\sin x) = 5\cos(25x^2) + \sin x\cos(\cos^2 x).$

Problem 62. If we are to apply the Riemann sum, we break up the unit interval $[0, 1]$ into n pieces, each of length $1/n$. We have the formula

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x.$$

It looks like we should take $f(x) = \sqrt{x}$ (and the right endpoint), for then we would have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}$$

which is what is given. Therefore the sum is

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg|_0^1 = \frac{2}{3}.$$

Problem 66. For (a), we break up the real line as the function is defined. For $x < 0$, $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$. If $0 \le x \le 1$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \frac{t^2}{2}\bigg|_0^x = \frac{x^2}{2}.$$

If $1 < x \le 2$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^1 t\,dt + \int_1^x (2-t)\,dt = g(1) + \left(2t - \frac{t^2}{2}\right)_1^x = \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2} = -\frac{x^2}{2} + 2x - 1.$$

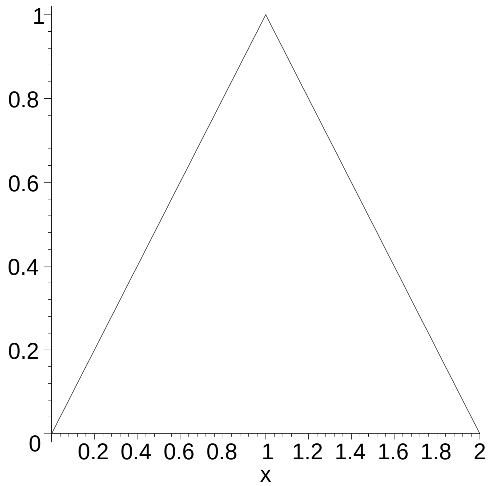
If $x > 2$, then

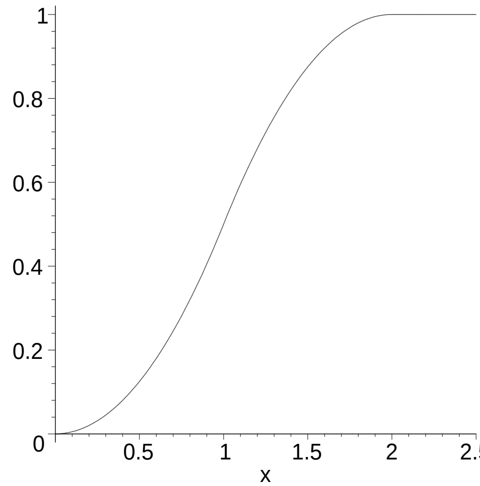
$$g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = -2 + 4 - 1 = 1.$$

Therefore

$$g(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{2}, & 0 \le x \le 1 \\ -\dfrac{x^2}{2} + 2x - 1, & 1 < x \le 2 \\ 1, & x > 2. \end{cases}$$

For (b), we have the following graphs of f and g, respectively:





Finally, for (c), we note that f is differentiable everywhere except at the corners, i.e. $x \neq 0, 1, 2$. However, g is differentiable everywhere, since $g'(x) = f(x)$ by the FTC!

5.4: Indefinite Integrals and the Net Change Theorem

Problem 2.

$$(x \sin x + \cos x + C)' = \sin x + x \cos x - \sin x = x \cos x.$$

Problem 4. Remembering that a is a constant and we are differentiating with respect to x, we have

$$\left(-\frac{(x^2+a^2)^{1/2}}{a^2x} + C\right)' = -\frac{a^2x(1/2)(x^2+a^2)^{-1/2}(2x) - (x^2+a^2)^{1/2}(a^2)}{a^4x^2} = -\frac{a^2x^2/\sqrt{x^2+a^2} - a^2\sqrt{x^2+a^2}}{a^4x^2}.$$

Now cancel a^2 from the top and bottom and simplify the numerator by multiplying top and bottom by $\sqrt{x^2+a^2}$:

$$-\frac{x^2 - (x^2+a^2)}{a^2x^2\sqrt{x^2+a^2}} = \frac{a^2}{a^2x^2\sqrt{x^2+a^2}} = \frac{1}{x^2\sqrt{x^2+a^2}}.$$

Problem 22.

$$\int_0^4 (2v+5)(3v-1)\,dv = \int_0^4 (6v^2 + 13v - 5)\,dv = (2v^3 + 13\frac{v^2}{2} - 5v)\Big|_0^4 = 128 + 104 - 20 = 212.$$

Problem 28.

$$\int_0^5 (2e^x + 4\cos x)\,dx = (2e^x + 4\sin x)\Big|_0^5 = 2e^5 + 4\sin 5 - 2 - 0 = 2e^5 + 4\sin 5 - 2.$$

Problem 38.

$$\int_4^9 (\sqrt{x} + 1/\sqrt{x})^2\,dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right)dx = \left(\frac{x^2}{2} + 2x + \ln|x|\right)_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \frac{85}{2} + \ln\frac{9}{4}.$$

Problem 40. Break up the integral to get rid of the absolute value. Note that $\sin x \geq 0$ on this interval from $[0, \pi]$ and $\sin x \leq 0$ on $[\pi, 3\pi/2]$. Therefore

$$\int_0^{3\pi/2} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} -\sin x\,dx = (-\cos x)_0^{\pi} + (\cos x)_{\pi}^{3\pi/2} = (1-(-1))+(0-(-1)) = 2+1 = 3.$$

Problem 44. Since $y = \sqrt[4]{x}$, we have $x = y^4$. We integrate $x = y^4$ from $y = 0$ to $y = 1$, so therefore

$$A = \int_0^1 y^4\,dy = \left.\frac{y^5}{5}\right|_0^1 = \frac{1}{5}.$$

Problem 48. By the Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$. This represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

Problem 54. For (a), we have displacement given by

$$\int_1^6 v(t)\,dt = \int_1^6 (t^2 - 2t - 8)\,dt = \left(\frac{t^3}{3} - t^2 - 8t\right)_1^6 = (72 - 36 - 48) - (1/3 - 1 - 8) = -\frac{10}{3}\text{ m}.$$

For (b), displacement is the absolute value of the distance travelled:

$$\int_1^6 |v(t)|\,dt = \int_1^6 |t^2 - 2t - 8|\,dt.$$

Now $t^2 - 2t - 8 = (t-4)(t+2)$ and this is ≥ 0 when $t > 4$ or $t < -2$ and ≤ 0 for t in the interval $[-2, 4]$. Therefore

$$\begin{aligned}\int_1^6 |t^2 - 2t - 8|\,dt &= \int_1^4 -(t^2 - 2t - 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt = -\left(\frac{t^3}{3} - t^2 - 8t\right)_1^4 + \left(\frac{t^3}{3} - t^2 - 8t\right)_4^6 \\ &= (-64/3 + 16 + 32) - (-1/3 + 1 + 8) + (72 - 36 - 48) - (64/3 - 16 - 32) = \frac{98}{3}\text{ m}.\end{aligned}$$

Problem 56. For (a), $v'(t) = a(t) = 2t + 3$, so $v(t) = t^2 + 3t + C$. Since $v(0) = C = -4$, $v(t) = t^2 + 3t - 4$.

For (b), we have the distance travelled (as in 54(b))

$$\begin{aligned}\int_0^3 |t^2 + 3t - 4|\,dt &= \int_0^3 |(t+4)(t-1)|\,dt = \int_0^1 -(t^2 + 3t - 4)\,dt + \int_1^3 (t^2 + 3t - 4)\,dt \\ &= -\left(\frac{t^3}{3} + 3\frac{t^2}{2} - 4t\right)_0^1 + \left(\frac{t^3}{3} + 3\frac{t^2}{2} - 4t\right)_1^3 \\ &= (-1/3 - 3/2 + 4) + (9 + 27/2 - 12) - (1/3 + 3/2 - 4) = \frac{89}{6}\text{ m}.\end{aligned}$$

Problem 58. By the Net Change Theorem, the amount of water that flows from the tank is

$$\int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = (200t - 2t^2)_0^{10} = 2000 - 200 = 1800\text{ liters}.$$

Yeehaw!