QUIZ #4: CALCULUS 1A (Stankova)
Wednesday, February 18, 2004
Section 10:00–11:00 (Voight)

Problem 1. *Find the derivative of the function*

$$F(x) = \frac{x^2 + 6\sqrt{x} + 5}{\sqrt{x}}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent.

Solution. By the Quotient Rule, we have

$$\begin{aligned} F'(x) &= \frac{ho\,dhi - hi\,dho}{ho^2} \\ &= \frac{\sqrt{x}\left(2x + \frac{3}{\sqrt{x}}\right) - (x^2 + 6\sqrt{x} + 5)\left(\frac{1}{2\sqrt{x}}\right)}{x} \\ &= \frac{1}{x}\left(2x\sqrt{x} + 3 - \frac{1}{2}x\sqrt{x} - 3 - \frac{5}{2\sqrt{x}}\right) \\ &= \frac{1}{x}\left(\frac{3}{2}x\sqrt{x} - \frac{5}{2\sqrt{x}}\right) = \frac{3}{2\sqrt{x}} - \frac{5}{2x\sqrt{x}}. \end{aligned}$$

Simplifying, we have

$$F(x) = x^{3/2} + 6 + 5x^{-1/2}$$

so by the Power Rule,

$$F'(x) = \frac{3}{2}x^{1/2} - \frac{5}{2}x^{-3/2}.$$

These are the same, since $x^{1/2} = \sqrt{x}$ and $x^{-3/2} = \dfrac{1}{x\sqrt{x}}$.

QUIZ #4: CALCULUS 1A (Stankova)
Wednesday, February 18, 2004
Section 11:00–12:00 (Voight)

Problem 1. *Find the derivative of the function*

$$f(x) = \frac{2}{x^2 - x}$$

using the definition of the derivative. State the domain of the function and the domain of the derivative.

You may not use only differentiation laws!

SOLUTION. By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{2}{(x+h)^2 - (x+h)} - \dfrac{2}{x^2 - x}}{h}. \end{aligned}$$

No need to simplify the denominators! We look for the common denominator (the product of both) and combine; the first factor is missing an $x^2 - x$ and the second is missing a $(x+h)^2 - (x+h)$, so we get

$$f'(x) = \lim_{h\to 0} \frac{2(x^2 - x) - 2((x+h)^2 - (x+h))}{h(x^2 - x)((x+h)^2 - (x+h))}.$$

Remember, the h enters the denominator since we are dividing by it. Now we multiply out the numerator and hope to cancel the h in the denominator. The numerator is

$$(2x^2 - 2x) - 2(x^2 + 2xh + h^2 - x - h) = -4xh - 2h^2 + 2h = h(-4x - 2h + 2),$$

so cancelling the h, we get

$$f'(x) = \lim_{h\to 0} \frac{-4x - 2h + 2}{h(x^2 - x)((x+h)^2 - (x+h))} = \frac{-4x + 2}{(x^2 - x)(x^2 - x)}.$$

The domain of the function is wherever the denominator does not vanish, $x^2 - x \neq 0$, i.e. $x \neq 0, 1$. The same is true of the derivative: its domain is $x \neq 0, 1$.

Check your work using the quotient rule:

$$f'(x) = \frac{(x^2 - x)(0) - 2(2x - 1)}{(x^2 - x)^2} = \frac{-4x + 2}{(x^2 - x)^2}.$$

Yup!

The algebra is also not so bad if you use the alternative definition:

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\dfrac{2}{x^2-x} - \dfrac{2}{a^2-a}}{x-a} \\
&= \lim_{x\to a} \frac{2(a^2-a)-2(x^2-x)}{(x-a)(x^2-x)(a^2-a)} \\
&= \lim_{x\to a} \frac{2a^2-2x^2+2x-2a}{(x-a)(x^2-x)(a^2-a)} \\
&= \lim_{x\to a} \frac{2(a-x)(a+x)+2(x-a)}{(x-a)(x^2-x)(a^2-a)} \\
&= \lim_{x\to a} \frac{-2(x+a)+2}{(x^2-x)(a^2-a)} = \frac{-4a+2}{(a^2-a)^2}.
\end{aligned}
$$