MATH 110: LINEAR ALGEBRA
HOMEWORK #4 WORKSHEET

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Work on exercise assigned to your group (by number) at the board, and then write up together a nice solution that everyone can understand. If you finish early, return to your desks and work together on another problem of your choice.

(1) [2.2.10] Let V be a vector space (over a field F). Let W_1, W_2 be subspaces such that $V = W_1 \oplus W_2$. Let $T : V \to V$ be a linear transformation such that $T(W_1) \subset W_1$ and $T(W_2) \subset W_2$. Prove that there is an ordered basis β of V such that $[T]_\beta$ has the form
$$\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix},$$
where A_1 and A_2 are matrices. What are the dimensions of the matrices A_1 and A_2? Prove that $T(W_1) = W_1$ if and only if A_1 is an invertible matrix.

(2) [2.2.14] Let $V = P(\mathbb{R})$, and for $j \geq 1$ define $T_j : V \to V$ by $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{T_1, T_2, \ldots, T_n\}$ is a linearly independent subset of $\mathcal{L}(V)$ for any positive integer n.

(3) [2.3.11] Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ (the zero transformation) if and only if $R(T) \subset N(T)$. Prove that if one of these two equivalent conditions holds, and V is finite-dimensional, then there is an ordered basis β for V such that $[T]_\beta$ has the form
$$\begin{pmatrix} O & A \\ O & O \end{pmatrix}$$
for some matrix A.

(4) [2.3.18(a)] Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear. If $\operatorname{rk}(T) = \operatorname{rk}(T^2)$, then prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

(5) [2.4.15] Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

(6) "Label" true or false:
   (a) Any two vector spaces of the same (finite) dimension are isomorphic.
   (b) The set of $n \times n$ invertible matrices is a subspace of $M_{n\times n}(F)$.
   (c) If A is a matrix, then $A^2 = A$ implies $A = I$ or $A = O$ (the identity and zero matrix, respectively).
   (d) If $T : V \to W$ is an invertible linear transformation, and β is a basis for V, γ a basis for W, then $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\beta^\gamma$.

(7) Let B be an $n \times n$ matrix. Prove that the map $T_B : M_{n\times n}(F) \to M_{n\times n}(F)$ by $A \mapsto AB$ is an isomorphism if and only if B is invertible.

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