MATH 110: LINEAR ALGEBRA
MIDTERM #1 REVIEW

Problem 1. Let $V = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

(a) $\mathbb{R}$ is a vector space over the rational numbers $\mathbb{Q}$. Prove that V is a subspace of $\mathbb{R}$ (over $\mathbb{Q}$).

(b) Show that $\beta = \{1, \sqrt{2}\}$ is a basis for V.

(c) Prove that the map $T : V \to V$ by $x \mapsto x\sqrt{2}$ is a linear transformation.

(d) Compute $[T]_\beta$.

(e) Prove that T is an isomorphism.

Problem 2. Let V be a finite-dimensional vector space, and let W be a subspace of V. Show that there exists a subspace $Z \subset V$ such that $V = W \oplus Z$.

Problem 3. Let V and W be finite-dimensional vector spaces over a field F, and let

$$V \times W = \{(v, w) : v \in V, w \in W\}.$$

Then $V \times W$ is a vector space over F, by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v, w) = (cv, cw)$$

for $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $c \in F$.

(a) What is the dimension of $V \times W$?

(b) Prove or disprove: $V \times W$ is isomorphic to $\mathcal{L}(V, W)$.

Problem 4. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $f(x) \mapsto f'(x)$. Let β be the ordered basis $1, x, x^2$ of $P_2(\mathbb{R})$, and let β' be the ordered basis $1, 1+x, 1+x+x^2$. Find a matrix Q such that

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Problem 5. Let $v_1 = (2, 1)$ and $v_2 = (1, -1)$. Then $\beta = v_1, v_2$ is an ordered basis for $V = \mathbb{R}^2$.

(a) Prove that there exist linear functionals $f_1, f_2 : V \to \mathbb{R}$ satisfying:

$$f_1(v_1) = 1, f_1(v_2) = 0; \quad f_2(v_1) = 0, f_2(v_2) = 1.$$

(b) Find a formula for $f_1(x, y)$.

(c) Prove that $\{f_1, f_2\}$ is a basis for V^*.