MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #2

Let R be a ring and let M be a (left) R-module.

Problem 1 (DF 10.1.1, 10.1.3).

(a) Prove that $0m = 0$ and $(-1)m = -m$ for all $m \in M$.
(b) Let $r \in R$ and suppose that $rm = 0$ for some nonzero $m \in M$. Prove that $r \notin R^\times$.

Problem 2 (DF 10.1.11). Let M be the abelian group (i.e., $\mathbb{Z}$-module) $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$.

(a) Find $\operatorname{Ann}(M)$, the annihilator of M in $\mathbb{Z}$.
(b) Let $I = 2\mathbb{Z}$. Describe the annihilator of I in M as a direct product of cyclic groups.

Problem 3 (DF 10.1.8, 10.2.8, 10.3.4). An element $m \in M$ is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements is denoted $\operatorname{Tor}(M)$.

(a) Prove that if R is an integral domain, then $\operatorname{Tor}(M)$ is a submodule of M.
(b) Give an example of a ring R and an R-module M such that $\operatorname{Tor}(M)$ is not a submodule. *[Hint: Consider the torsion elements in $M = R$.]*
(c) Show that if R is not an integral domain, then every nonzero R-module M has $\operatorname{Tor}(M) \neq \{0\}$.
(d) Let $\phi : M \to N$ be an R-module homomorphism. Prove that $\phi(\operatorname{Tor}(M)) \subset \operatorname{Tor}(N)$.
(e) M is called a *torsion module* if $M = \operatorname{Tor}(M)$. Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

Problem 4 (sorta DF 10.1.19). Let $V = \mathbb{R}^2$, and let $T : V \to V$ be the linear transformation which is projection onto the y-axis. Show that the only submodules of the $\mathbb{R}[x]$-module corresponding to T are V, the x-axis, the y-axis, and $\{(0,0)\}$.

Problem 5 (DF 10.2.6). Describe the $\mathbb{Z}$-module $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/21\mathbb{Z}, \mathbb{Z}/30\mathbb{Z})$.

Problem 6 (sorta DF 10.2.7). Let R be commutative. Show that the map $R \to \operatorname{End}_R(M)$ where $r \in R$ maps to the multiplication-by-r endomorphism

$$\begin{aligned} \phi_r : M &\to M \\ m &\mapsto rm \end{aligned}$$

is a ring homomorphism.

Problem 7 (DF 10.2.9–10). Let R be commutative.

(a) Prove that $\operatorname{Hom}_R(R, M) \cong M$ as R-modules. *[Hint: Show that each element of* $\operatorname{Hom}_R(R, M)$ *is determined by its value on* $1 \in R$.*]*
(b) Prove that $\operatorname{End}_R(R) \cong R$ as rings.

Problem 8. Let $\phi : M \to M$ be an R-module homomorphism such that $\phi \circ \phi = \phi$. Show that

$$M = \ker \phi \oplus \operatorname{img} \phi.$$

Problem 9 (DF 10.3.7). Let N be a submodule of M. Prove that if both M/N and N are finitely generated, then so is M.

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