MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #4A

Problem 1 (DF 12.1.2). Let R be an integral domain and let M be an R-module. The *rank* of M is the maximal number of R-linearly independent elements of M.

(a) Suppose that M has rank n and that $x_1, \dots, x_n$ is any maximal set of R-linearly indepedent elements of M. Let $N = Rx_1 + \dots + Rx_n$ be the R-submodule generated by $x_1, \dots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module. *[Hint: Show that the map $R^n \to M$ which sends the ith standard basis vector to x_i is an isomorphism of R-modules.]*

(b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \cong R^n$) such that the quotient M/N is a torsion R-module then M has rank n. *[Hint: Let $y_1, \dots, y_{n+1}$ be any $n+1$ elements of M. Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements r_i of R. Use an argument like Proposition 12.1.3 to show that the $r_i y_i$, and hence also the y_i, are linearly dependent.]*

Problem 2 (DF 12.1.5). Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and x, considered as a submodule of R. Show that $\{2, x\}$ is not a basis of M. *[Hint: Find a nontrivial R-linear dependence between these two elements.]* Show that the rank of M is 1 but that M is not free of rank 1.

Problem 3 (DF 12.1.13). If M is a finitely generated module over a PID, describe the structure of $M/\operatorname{Tor}(M)$.

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