MATH 252: ABSTRACT ALGEBRA II
WORKSHEET, DAY #12

Fill in the blanks.

Let F be a field. An F-module V is also known as a ___F-vector space___ over F, and an F-module homomorphism $\phi : V \to W$ is called a ____________________ .

Let V be a vector space over F. Let $v_1, \dots, v_n \in V$. A *linear combination* of $v_1, \dots, v_n$ is

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The set of all vectors w which are linear combinations of $v_1, \dots, v_n$ forms a ____________________ $W \subset V$, and we say that W is ____________________ by $v_1, \dots, v_n$.

A *linear relation* among vectors $v_1, \dots, v_n$ is a linear combination which is equal to zero, i.e.,

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The vectors $v_1, \dots, v_n$ are called ____________________ if there is no nonzero linear relation among the vectors, i.e., if $c_1 v_1 + \dots + c_n v_n = 0$ then ____________________________________ ; otherwise $v_1, \dots, v_n$ are called ____________________ . By convention, the

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empty set is considered to be ______________________ , and the span of the empty set is ______________________ . Two vectors v_1, v_2 have no nonzero linear relation if and only if either

______________________ or ____________________________ .

An ordered set $B = \{v_1, \ldots, v_n\}$ of vectors which is linearly independent and spans V is called a ______________________ of V; for example, for $F^n = \{(a_1, \ldots, a_n) : a_i \in F\}$ we may take

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Lemma. *The set B is a basis for V if and only if every $w \in V$ can be written uniquely as a*

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Proposition. *Let $L = \{v_1, \ldots, v_n\} \subset V$ be a linearly independent ordered set, and let $v \in V$. Then the ordered set $\{v_1, \ldots, v_n, v\}$ is linearly independent if and only if*

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Proposition. *For any finite set S which spans V, there exists a subset $B \subset S$ which is a basis for V.*

Proof. Suppose that $S = \{v_1, \ldots, v_n\}$ and that S is not linearly independent. Then

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Lemma. *Let V be a vector space with a finite basis. Then any spanning set of V contains a basis, and any* ____________________ *set L can be extended by adding elements of V to get a basis.*

Corollary. *Suppose V has a finite basis B with $\#B = n$. Then any set of linearly independent vectors has at most* ________ *elements, and any spanning set has* ________________ *elements.*

Proof. Let L be a linearly independent set of vectors. By the lemma,

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Corollary. *If V has a finite basis then any two bases of V have the same cardinality.*

Proof. ____________________________________

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Let V be a vector space with a finite basis. Then the __dimension__ of V is defined to be ____________________________ and is denoted $\dim_F V$, and V is said to be ____________________ over F.

If F is a finite field with $\#F = q$, then a vector space of dimension n over F has __________ elements.

Theorem. *Let V be a vector space of dimension n. Then $V \cong F^n$. In particular, any two vector spaces of the same finite dimension are isomorphic.*

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Define the map

$$\phi : F^n \to V$$

$$\phi(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

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Theorem. *Let V be a finite-dimensional vector space over F and let W be a subspace of V. Then the quotient V/W is a vector space with*

$\dim(V/W) =$ ______________________________ , *Proof.* Since V is finite-dimensional, so is W because

______________________________ . So let W have dimension m and let $w_1, \ldots, w_m$ be a basis for W. We extend this basis to a basis $w_1, \ldots, w_m, v_{m+1}, \ldots, v_n$ of V. Then the projection map $V \to V/W$ maps each w_i to __________ and therefore has image spanned by $v_{m+1}+W, \ldots, v_n+W$; these vectors are linearly independent because

____ . So

$\dim(V/W) =$ ____________________ . □

Corollary. *Let $\phi : V \to W$ be a linear transformation. Then*

$$\dim V = \dim \ker \phi + \dim \operatorname{img} \phi.$$

We also say that $\ker \phi$ is the ____________________ of ϕ and $\dim \ker \phi$ is the ____________________ . The dimension of $\operatorname{img} \phi = \phi(V)$ is called the ____________________ .

Corollary. *Let $\phi : V \to W$ be a linear transformation of vector spaces of the same finite dimension n. Then ϕ is an isomorphism if and only if ϕ is injective if and only if ϕ is surjective.*

Proof. ____

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