MATH 255: ELEMENTARY NUMBER THEORY HOMEWORK #4

JOHN VOIGHT

3.7: Linear Diophantine Equations

Problem 3.7.1. For each of the following linear diophantine equations, either find all solutions, or show that there are no integral solutions.

(a) $2x + 5y = 11$.
(b) $17x + 13y = 100$.
(d) $60x + 18y = 97$.

Problem 3.7.18. Is it possible to have 50 coins, all of which are pennies, dimes, or quarters, with a total worth \$3? *[Hint: Eliminate variables.]*

Problem 3.7.19–20*. The following problem is colloquially known as the *postage stamp problem*. For bureaucratic reasons, the Postal Service decides that it will now print only two stamps. Outraged, an angry mob insists that this will make mailing packages impossible.

Show that the angry mob is wrong. Let $a, b \in \mathbb{Z}_{>0}$ be relatively prime. We say that $n \in \mathbb{Z}$ is a *nonnegative* linear combination of a, b if there exists $x, y \in \mathbb{Z}_{\geq 0}$ such that $n = ax + by$. Show that $ab - a - b$ cannot be written as a nonnegative linear combination of a, b, but every $n > ab - a - b$ can. Conclude that every sufficiently large postage can be obtained with only stamps with denominations a and b.

4.1: Introduction to Congruences

Problem 4.1.4. Show that if a is an even integer, then $a^2 \equiv 0 \pmod 4$, and if a is an odd integer, then $a^2 \equiv 1 \pmod 4$.

Problem 4.1.8. Show that if a, b, m, n are integers such that $m, n > 0$ and $n \mid m$, then $a \equiv b \pmod m$ implies $a \equiv b \pmod n$.

Problem 4.1.16. Which decimal digits occur as the final digit of a fourth power of an integer?

Problem 4.1.17. What can you conclude if $a^2 \equiv b^2 \pmod p$, where a and b are integers and p is prime?

4.2: Linear Congruences

Problem 4.2.1. Find all solutions of each of the following linear congruences.

(a) $2x \equiv 5 \pmod 7$.
(b) $3x \equiv 6 \pmod 9$.
(c) $19x \equiv 30 \pmod{40}$.

Date: Due Wednesday, 11 February 2009.

Problem 4.2.6. For which integers c, $0 \leq c < 30$, does the congruence $12x \equiv c \pmod{30}$ have solutions? When there are solutions, how many incongruent solutions are there?