MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #1

Problem 1.

(a) Let $a = 3 - 8i$ and $b = 2 + 3i$. Find $x, y \in \mathbb{Z}[i]$ such that $ax + by = 1$.

(b) Show explicitly that the ideal $I = (85, 1 + 13i) \subseteq \mathbb{Z}[i]$ is principal by exhibiting a generator.

Problem 2 (sorta DF 8.1.8). Show that the ring $\mathbb{Z}[\rho]$ where $\rho = (-1 + \sqrt{-3})/2$ is a Euclidean domain. *[Hint: Plot the points of $\mathbb{Z}[\rho]$ in $\mathbb{C}$ and mimic the proof for $\mathbb{Z}[i]$.]*

Problem 3 (DF 8.1.3). Let R be a Euclidean domain. Let m be the smallest (nonnegative) integer in the set $\{N(a) : 0 \neq a \in R\}$.

(a) Prove that every nonzero element of R of norm m is a unit.

(b) Deduce that a nonzero element of norm zero is a unit, and show by example that the converse of this statement is false.

Problem 4* (DF 8.1.6). The following problem is colloquially known as the *postage stamp problem*. For bureaucratic reasons, the Postal Service decides that it will now print only two stamps. Outraged, an angry mob insists that this will make mailing packages impossible.

Show that the angry mob is wrong. Let $a, b \in \mathbb{Z}_{>0}$ be relatively prime. We say that $N \in \mathbb{Z}$ is a *linear combination* of a, b if there exists $x, y \in \mathbb{Z}$ such that $N = ax + by$, and we say that N is a *nonnegative* linear combination if $x, y \in \mathbb{Z}_{\geq 0}$. Show that $ab - a - b$ cannot be written as a nonnegative linear combination of a, b, but every $N > ab - a - b$ can. Conclude that every sufficiently large postage can be obtained with only stamps with denominations a and b.

Date: 20 January 2012; due Friday, 27 January 2012.