MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #3

Problem 1. Show that $(2, x)$ is a maximal ideal but not a principal ideal in $\mathbb{Z}[x]$.

Problem 2 (DF 9.2.1–3). Let F be a field and let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$.

(a) Let $^{-}$ denote passage to the quotient $F(x)/(f(x))$. Prove that for each $\overline{g(x)}$, there exists a unique polynomial $\overline{r(x)}$ of degree $\leq n-1$ such that $\overline{g(x)} = \overline{r(x)}$. *[Hint: Use the division algorithm.]*
(b) Suppose $\#F = q$. Show that $\#F(x)/(f(x)) = q^n$.
(c) Show that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. *[Hint: Use* Proposition 7, Section 8.2.*]*

Problem 3 (sorta DF 9.4.20). Here we see some pathologies in $R[x]$ when R is a ring but not an integral domain.

(a) Show that in $\mathbb{Z}/6\mathbb{Z}[x]$, the polynomial x factors as $x = (3x + 4)(4x + 3)$.
(b) Show that the ideal $(3, x)$ is a principal ideal in $\mathbb{Z}/6\mathbb{Z}[x]$.

Problem 4* (sorta DF 9.1.14). Let R be an integral domain, and let $S = R[x, y]$.

(a) Prove that the ideal $(x^4 - y^2)$ is not a prime ideal in S.
(b) Prove that the ideal $(x^3 - y^2)$ is a prime ideal in S. *[Hint: Consider the ring homomorphism $\phi : R[x, y] \to R[t]$ with $x, y \mapsto t^2, t^3$.]*

Date: 3 February 2012; due Friday, 10 February 2012.