MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #4

Let R be a ring and let M be a (left) R-module.

Problem 1 (DF 10.1.3, 10.1.6).

(a) Let $r \in R$ and suppose that $rm = 0$ for some nonzero $m \in M$. Prove that $r \notin R^\times$.

(b) Show that the intersection of any nonempty collection of submodules of an R-module is a submodule.

Problem 2 (DF 10.1.11). Let M be the $\mathbb{Z}$-module $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$.

(a) Find $\operatorname{ann}(M)$, the annihilator of M in $\mathbb{Z}$.

(b) Let $I = 2\mathbb{Z}$. Describe the annihilator of I in M as a direct product of cyclic groups.

Problem 3 (sorta DF 10.1.19). Let $V = \mathbb{R}^2$, and let $T : V \to V$ be the linear transformation which is projection onto the y-axis. Show that the only submodules of the $\mathbb{R}[x]$-module corresponding to T are V, the x-axis, the y-axis, and $\{(0,0)\}$.

Problem 4* (DF 10.1.8). An element $m \in M$ is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements is denoted $\operatorname{Tor}(M)$.

(a) Prove that if R is an integral domain, then $\operatorname{Tor}(M)$ is a submodule of M.

(b) Give an example of a ring R and an R-module M such that $\operatorname{Tor}(M)$ is not a submodule.

(c) Show that if R has a zerodivisor then every nonzero R-module M has $\operatorname{Tor}(M) \neq \{0\}$.

(d) M is called a *torsion module* if $M = \operatorname{Tor}(M)$. Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

Date: 10 February 2012; due Friday, 17 February 2012.