MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #5

Let R be a ring and let M be a (left) R-module.

Problem 1 (DF 10.2.6). Describe the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/21\mathbb{Z}, \mathbb{Z}/30\mathbb{Z})$.

Problem 2 (sorta DF 10.2.7). Let R be commutative. Show that the map $R \to \mathrm{End}_R(M)$ where $r \in R$ maps to the multiplication-by-r endomorphism

$$\begin{aligned} \phi_r : M &\to M \\ m &\mapsto rm \end{aligned}$$

is a ring homomorphism, and thereby that $\mathrm{End}_R(M)$ has the structure of an R-algebra.

Problem 3 (DF 10.2.9–10). Let R be commutative.

(a) Prove that $\mathrm{Hom}_R(R, M) \cong M$ as R-modules. *[Hint: Show that each element of $\mathrm{Hom}_R(R, M)$ is determined by its value on $1 \in R$.]*

(b) Consider R as an R-module. Prove that $\mathrm{End}_R(R) \cong R$ as rings.

Problem 4. Let $\phi : M \to M$ be an R-module homomorphism such that $\phi \circ \phi = \phi$. Show that

$$M = \ker \phi \oplus \operatorname{img} \phi.$$

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