MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #6

Problem 1 (DF 10.3.7). Let N be a submodule of M. Prove that if both M/N and N are finitely generated, then so is M.

Problem 2 (DF 11.1.8, 11.1.9, 11.2.8). Let V be a vector space over F and let $\phi : V \to V$ be a linear transformation. A nonzero element $v \in V$ satisfying $\phi(v) = \lambda v$ for some $\lambda \in F$ is called an *eigenvector* of ϕ with *eigenvalue* λ.

(a) Prove that for any fixed $\lambda \in F$, the collection of eigenvectors of ϕ with eigenvalue λ, together with 0, forms a subspace of V.

(b) Suppose for $i = 1, \dots, k$ that $v_i \in V$ is an eigenvector of ϕ with eigenvalue λ_i and that all of the eigenvalues λ_i are distinct. Prove that $v_1, \dots, v_k$ are linearly independent. Conclude that any linear transformation on an n-dimensional vector space has at most n distinct eigenvalues.

(c) Prove that if V has a basis consisting of eigenvectors of ϕ, then the matrix representing ϕ with respect to this basis is diagonal. What are the diagonal entries?

(d) Prove that an $n \times n$ matrix A is similar to a diagonal matrix if and only if F^n has a basis of eigenvectors for the linear transformation

$$\begin{aligned} L(A) : F^n &\to F^n \\ x &\mapsto Ax. \end{aligned}$$

Problem 3. Let $\phi : V \to V$ be a linear transformation over a field F and β a basis of V.

(a) Show that ϕ is invertible if and only if ϕ maps β to a basis of V if and only if the column vectors of $M(\phi)_\beta^\beta$ are a basis of V.

(b) Suppose that $\#F = q$. Show that

$$\#GL_n(F) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

Problem 4 (sorta DF 11.2.35).

(a) Define the *trace* map

$$\begin{aligned} \mathrm{tr} : M_2(F) &\to F \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto a + d. \end{aligned}$$

Show that tr is a linear transformation and determine the matrix of tr with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_2(F)$.

(b) Generalize part (a) to $\mathrm{tr} : M_n(F) \to F$ for arbitrary $n \in \mathbb{Z}_{>0}$.

Date: 2 March 2012; due Friday, 16 March 2012.

Problem 5* (sorta DF 11.1.5). Let $a, b \in \mathbb{R}$ with $a < b$. Let V denote the space of real-valued functions on the closed interval $[a, b]$.

(a) Show that V is isomorphic to an uncountably infinite direct product of copies of $\mathbb{R}$.

(b) Let $C([a, b]) \subset V$ denote the subspace of continuous functions. Show that for any $g \in C([a, b])$, the function $\phi_g : V \to \mathbb{R}$ defined by $\phi_g(f) = \int_a^b f(t)g(t)\,dt$ is a linear functional on $C([a, b])$.

(c) Let

$$W = \mathbb{R}[x]_{\leq 2} = \{a_2x^2 + a_1x + a_0 : a_i \in \mathbb{R}\} \subset C([a, b]).$$

Let $\beta = \{1, x, x^2\}$. For each $f^* \in \beta^* \subset W^*$, find a $g(x) \in W$ such that $f^* = \phi_g$.

Problem 6* (DF 11.3.4–5). Let V be a vector space with basis β.

(a) Show that V^* is isomorphic to the direct product of copies of F indexed by β.

(b) If $\#\beta = \infty$, show that β^* does *not* span V^*, hence $\dim V^* > \dim V$.