MATH 252 HOMEWORK TEMPLATE

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Problem 7.3.2. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Solution. Suppose $\phi : \mathbb{Q}[x] \to \mathbb{Z}[x]$ is a ring homomorphism. Then since $\phi(1) = 1$ by definition of a ring homomorphism,

$$\phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 1 + 1 = 2.$$

But now

$$1 = \phi(1) = \phi(2 \cdot 1/2) = 2 \cdot \phi(1/2)$$

Thus $\phi(1/2) \in \mathbb{Z}[x]^\times$, and $\mathbb{Z}[x]^\times = \mathbb{Z}^\times = \{\pm 1\}$ by Proposition 7.2.4. Since $2 \cdot \pm 1 \neq 1$, we have a contradiction; so no such homomorphism ϕ exists.

Problem 7.3.11. Let R be the ring of all continuous real-valued functions on the closed interval $[0, 1]$. Prove that the map $\phi : R \to \mathbb{R}$ defined by $\phi(f) = \int_0^1 f(t)\,dt$ is a homomorphism of additive groups but not a ring homomorphism.

Solution. If $f, g \in R$, we have

$$\phi(f+g) = \int_0^1 (f+g)(t)\,dt = \int_0^1 (f(t)+g(t))\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = \phi(f) + \phi(g)$$

so ϕ is a homomorphism of additive groups. It is not, however, a ring homomorphism: if $f : [0, 1] \to \mathbb{R}$ is the inclusion map $f(x) = x$, then we have

$$\phi(f^2) = \int_0^1 t^2\,dt = \frac{1}{3} \neq \frac{1}{2}^2 = \left(\int_0^1 t\,dt\right)^2 = \phi(f)^2.$$

Problem 7.3.18.

(a) If I and J are ideals of R prove that their intersection $I \cap J$ is also an ideal of R.

Solution. First, I and J are subgroups of R under $+$ so $I \cap J$ is also subgroup of R by Exercise 2.1.10(a).

Next, we show that $I \cap J$ is closed under multiplication by R. Let $x \in I \cap J$ and $r \in R$. Then $x \in I$ and I is an ideal so $rx, xr \in I$, and similarly $rx, xr \in J$; thus $rx, xr \in I \cap J$.

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