Alternate Mirror Families and Hypergeometric Motives

Charles Doran, Tyler Kelly, Adriana Salerno, Steven Sperber, John Voight, and Ursula Whitcher

Abstract Mirror symmetry predicts surprising geometric correspondences between distinct families of algebraic varieties. In some cases, these correspondences have arithmetic consequences. Among the arithmetic correspondences predicted by mirror symmetry are correspondences between point counts over finite fields, and more generally between factors of their Zeta functions. In particular, we will discuss our results on a common factor for Zeta functions of alternate families of invertible polynomials. We will also explore closed formulas for the point counts for our alternate mirror families of K3 surfaces and their relation to their Picard–Fuchs equations. Finally, we will discuss how all of this relates to hypergeometric motives. This report summarizes work from two preprints in progress.

Charles F. Doran
University of Alberta, Department of Mathematics, Edmonton, AB Canada, e-mail: doran@math.ualberta.edu

Tyler L. Kelly
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK, e-mail: tlk20@dpmms.cam.ac.uk

Adriana Salerno
Department of Mathematics, Bates College, 3 Andrews Rd, Lewiston, ME 04240, USA, e-mail: asalerno@bates.edu

Steven Sperber
School of Mathematics, University of Minnesota, 206 Church Street SE, Minneapolis, MN 55455 USA, e-mail: sperber@umn.edu

John Voight
Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA, e-mail: jvoight@gmail.com

Ursula Whitcher
Mathematical Reviews, 416 Fourth St, Ann Arbor, MI 48103, USA, e-mail: uaw@umich.edu

1 Motivation

Calabi–Yau varieties—those smooth projective varieties with trivial canonical bundle—provide a rich and interesting source of arithmetic and geometry. Calabi–Yau varieties of dimension 1 are elliptic curves, ubiquitous in mathematics and theoretical physics. In dimensions two and above, we take our Calabi–Yau varieties to be simply connected. The two-dimensional Calabi–Yau varieties are better known as *K3 surfaces*, after the mathematicians Kummer, Kähler, and Kodaira and the mountain K2. Like elliptic curves, K3 surfaces are all diffeomorphic to each other, but the study of their complex and arithmetic structure remains deep. The study of higher dimensional Calabi–Yau varieties promise the same rewards in many areas of mathematics.

It is particularly important to study Calabi–Yau varieties in families, and interesting families of Calabi–Yau varieties arise in several ways. Perhaps the simplest method of obtaining Calabi–Yau varieties is to take smooth $(n+1)$-folds in projective space $\mathbb{P}^n$. A natural generalization of this construction is to take anticanonical hypersurfaces or complete intersections in certain toric varieties. Often, however, one wishes to consider subfamilies with further special properties. For example, a general smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ has Picard rank 1, but a general member of the pencil of K3 surfaces given by

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4\psi x_0 x_1 x_2 x_3 = 0$$

has Picard rank 19 and the Fermat quartic

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

where $\psi = 0$ has Picard rank 20. As often happens, these special geometric properties are correlated with enhanced symmetry: each member of the pencil admits an action by the group $(\mathbb{Z}/4\mathbb{Z})^2$, and the Fermat quartic admits an action by a group of 384 elements.

Calabi–Yau manifolds are also interesting from a physical perspective. Indeed, string theory posits that our universe consists of four space-time dimensions together with six extra, compact real dimensions which take the shape of a Calabi–Yau variety. Physicists have produced several consistent candidate theories, using properties of the underlying varieties. These theories are linked by *dualities* which transform physical observables described by one collection of geometric data into equivalent observables described by different geometric data. Attempts to build a mathematically consistent description of the duality between Type IIA and Type IIB string theories led to the thriving field of *mirror symmetry*, which is based on the philosophy that the complex moduli of a given family of Calabi–Yau varieties should correspond to the complexified Kähler moduli of a mirror family.

There are several methods of constructing the mirror to a family of Calabi–Yau varieties. The first mirror symmetry construction, due to Greene–Plesser [20], used a $(\mathbb{Z}/5\mathbb{Z})^3$ action on the one-parameter family of quintic threefolds X_ψ given by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0$$

to construct the mirror family Y_ψ to all smooth quintic hypersurfaces in $\mathbb{P}^4$. A directly analogous construction can be used to find the mirrors to families of Calabi–Yau hypersurfaces in weighted projective spaces. Batyrev gave combinatorial methods for constructing mirror families to Calabi–Yau varieties realized as hypersurfaces or complete intersection in toric varieties [3]. Though powerful, Batyrev's construction relates families rather than individual varieties. In the current work, we use an alternative generalization of the Greene–Plesser construction due to Berglund–Hübsch–Krawitz [4, 28], allowing for a direct comparison of varieties on either side of the mirror correspondence.

When individual pairs of mirror varieties can be identified, mirror symmetry constructions have implications for their arithmetic and geometric structure. These implications were first explored by Candelas–de la Ossa–Rodriguez-Villegas [6] for their zeta functions, the generating function for the number of $\mathbb{F}_{p^r}$-valued points

$$Z(X,T) = \exp\left(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{p^r})T^r}{r}\right)$$

for a variety X over $\mathbb{F}_p$; we have $Z(X,T) \in \mathbb{Q}(T)$ by a theorem of Dwork [14]. These authors used the Greene–Plesser mirror construction and techniques from toric varieties to compare the zeta function of fibers of the diagonal Fermat pencil of threefolds X_ψ and the mirror pencil of threefolds Y_ψ [6, 7, 5]. They found that for general ψ, the zeta functions of X_ψ and Y_ψ share a common factor $R(T,\psi)$. This common factor is related to the period of the holomorphic form on X_ψ, and the number of points on X_ψ over a finite field is given by a truncation of a generalized hypergeometric function which solves the Picard–Fuchs equation associated to the holomorphic form. Furthermore, the other nontrivial factors of $Z(X_\psi, T)$ were closely related to the action of $(\mathbb{Z}/5\mathbb{Z})^3$ on homogeneous monomials.

The Greene–Plesser construction generalizes easily to smooth hypersurfaces of degree $n+1$ in $\mathbb{P}^n$. Wan [34] has characterized the relationship between a member X_ψ of the diagonal Fermat pencil in $\mathbb{P}^n$ and its mirror Y_ψ in terms of point counts via the congruence

$$\#X_\psi(\mathbb{F}_q) \equiv \#Y_\psi(\mathbb{F}_q) \pmod{q}$$

for all $q = p^r$ such that $\mathbb{F}_q \supseteq \mathbb{F}_p(\psi)$. Fu–Wan [16] generalized this result to other pairs of mirror pencils. More recently, Kloosterman [26] showed that one can use a group action to describe the distinct factors of the zeta function for any one-parameter monomial deformation of a diagonal hypersurface in weighted projective space.

In our work, we take a slightly different approach. Rather than relating a pencil of Calabi–Yau varieties to its mirror, we instead consider those pencils whose mirrors are related in some geometric way. In other words, we seek to understand when common properties of mirrors translate into arithmetic, geometric, or physical implications for the original pencils themselves.

There is an intricate relationship between Picard–Fuchs equations and the zeta function, mediated by the action of the Frobenius map. Given a set of symmetric pencils in $\mathbb{P}^n$ which yield alternate mirrors to smooth $n+1$-folds in $\mathbb{P}^n$, we hypothesize that the zeta functions of the members of each pencil and their mirror should share a common factor, corresponding to the Picard–Fuchs equation satisfied by the holomorphic form. In the current work, we apply the formalism of Berglund–Hübsch–Krawitz mirror symmetry to characterize appropriate symmetric pencils, and we study the resulting zeta functions.

We have followed four approaches, exploiting algebraic, geometric, and arithmetic properties of highly symmetric pencils.

2 Common Factor Theorem

Our first result, described in more detail in [12], is that invertible pencils whose mirrors have common properties share arithmetic similarities as well. Revisiting work of Gährs [18], we find that invertible pencils whose BHK mirrors are hypersurfaces in quotients of the same weighted-projective space have the same Picard–Fuchs equation associated to their holomorphic form. In turn, we show that the Picard–Fuchs equations for the pencil dictate a factor of the zeta functions of the pencil.

An *invertible polynomial* is a polynomial

$$F_A = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}} \in \mathbb{Z}[x_0, \dots, x_n],$$

where $A = (a_{ij})_{i,j}$ is an $(n+1) \times (n+1)$ is a matrix with nonnegative integer entries, such that:

- $\det(A) \neq 0$,
- the polynomial F_A is homogeneous of degree $n+1$, and
- the function $F_A : \mathbb{C}^{n+1} \to \mathbb{C}$ has exactly one singular point at the origin.

We further impose that these hypersurfaces are Calabi–Yau varieties, so the degree of the polynomial F_A is $n+1$.

Inspired by Berglund-Hübsch-Krawitz (BHK) mirror symmetry, we look at the weights of the transposed polynomial

$$F_{A^T} := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ji}},$$

which will be a quasihomogeneous polynomial, i.e., there exist nonnegative integral weights $q_0, \dots, q_n$ so that $\gcd(q_0, \dots, q_n) = 1$ and F_{A^T} defines a hypersurface X_{A^T} in the weighted-projective space $W\mathbb{P}^n(q_0, \dots, q_n)$. We call $q_0, \dots, q_n$ the *dual weights* of F_A. Let $d^T = \sum_i q_i$ be the sum of the weights.

Using the dual weights, we define a one-parameter deformation of our invertible pencil. Consider the polynomials

$$F_{A,\psi} = \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}} - d^T \psi x_0 \cdots x_n \in \mathbb{Z}[\psi][x_0, \ldots, x_n].$$

We then have a family of hypersurfaces $X_{A,\psi} := Z(F_{A,\psi}) \subset \mathbb{P}^n$ in the parameter ψ, which we call an *invertible pencil.*

The Picard–Fuchs equation for the family $X_{A,\psi}$ is determined completely by the dual weights by work of Gährs [18, Theorem 3.6]. Indeed, Gährs computes the order of the Picard–Fuchs equation in terms of the q_i. There is an explicit formula for the order $D(\mathbf{q})$ of the Picard–Fuchs equation that depends solely on the $(n+1)$-tuple of dual weights $\mathbf{q} = (q_0, \ldots, q_n)$. The Picard–Fuchs equation itself depends solely on $\mathbf{q}$ as well. To be precise, we observe that the Picard–Fuchs equation is a hypergeometric differential equation whose motive descends to $\mathbb{Q}$.

For a smooth projective hypersurface X in $\mathbb{P}^n$, the zeta function is of the form

$$Z(X,T) = \frac{P_X(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

with $P_X(T) \in \mathbb{Q}[T]$. Our main result exhibits a (fiber-wise) common factor of the zeta function in the general setting suggested above.

Theorem 1. *Let $X_{A,\psi}$ and $X_{B,\psi}$ be invertible pencils of Calabi–Yau $n-1$-folds in $\mathbb{P}^n$, determined by integer matrices A and B, respectively. Suppose A and B have the same dual weights q_i. Then for each $\psi \in \mathbb{F}_q$ such that the fibers $X_{A,\psi}$ and $X_{B,\psi}$ are smooth and $\gcd(q,(n+1)d^T) = 1$, the polynomials $P_{X_{A,\psi}}(T)$ and $P_{X_{B,\psi}}(T)$ have a common factor $R_\psi(T) \in \mathbb{Q}[T]$ with $\deg R_\psi(T) \geq D(\mathbf{q})$.*

3 Explicit Computations and Hypergeometric Motives

We next focus our attention on invertible families of K3 surfaces, with dual weights $(1,1,1,1)$, which are as follows:

	Quartic Family	Symmetries
F_4	$x_0^4+x_1^4+x_2^4+x_3^4-4\psi x_0x_1x_2x_3=0$	$(\mathbb{Z}/4\mathbb{Z})^2$
$\mathsf{F}_1\mathsf{L}_3$	$x_0^4+x_1^3x_2+x_2^3x_3+x_3^3x_1-4\psi x_0x_1x_2x_3=0$	$\mathbb{Z}/7\mathbb{Z}$
$\mathsf{F}_2\mathsf{L}_2$	$x_0^4+x_1^4+x_2^3x_3+x_3^3x_2-4\psi x_0x_1x_2x_3=0$	$\mathbb{Z}/8\mathbb{Z}$
$\mathsf{L}_2\mathsf{L}_2$	$x_0^3x_1+x_1^3x_0+x_2^3x_3+x_3^3x_2-4\psi x_0x_1x_2x_3=0$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\subset(\mathbb{Z}/8\mathbb{Z})^2$
L_4	$x_0^3x_1+x_1^3x_2+x_2^3x_3+x_3^3x_0-4\psi x_0x_1x_2x_3=0$	$\mathbb{Z}/5\mathbb{Z}$

In [13] we analyze the zeta functions of the families given in Table 3. Using a classical viewpoint, we find that the hypergeometricity of the Picard–Fuchs equations associated to the five families predicts a motivic decomposition of the point counts over finite fields for our families. We see that the hypergeometric Picard–Fuchs equations for the primitive middle cohomology of the five families correspond to nontrivial hypergeometric summands in the point counts over finite fields. The core of this paper is the following theorem:

Theorem 2. *Let $\diamond \in \mathscr{F} = \{\mathsf{F}_4, \mathsf{F}_2\mathsf{L}_2, \mathsf{F}_1\mathsf{L}_3, \mathsf{L}_2\mathsf{L}_2, \mathsf{L}_4\}$ signify one of the five* K3 *families in Table* 3. *There is a canonical decomposition of the finite field point count for $N_{\mathbb{F}_q}(X_{\diamond,\psi})$ whose summands are either trivial or hypergeometric. Moreover, there exists an element in $H^2_{\text{prim}}(X_{\diamond,\psi})$ that satisfies a hypergeometric Picard–Fuchs differential equation with parameters $\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1}$ if and only if there exists a nontrivial summand in the canonical finite field point count $N_{\mathbb{F}_q}(X_{\diamond,\psi})$ corresponding to the hypergeometric function defined over $\mathbb{F}_q$ with parameters $\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1}$.*

This proof is done explicitly. First, we find the Picard–Fuchs equations via the diagrammatic method introduced in [6, 7] and fully developed in [11]. After establishing the hypergeometric forms of the Picard–Fuchs equations, we confirm that they do indeed correspond to those in the finite point counts using Gauss sums, using a classical method due to Delsarte [10] and Furtado Gomida [17].

Additionally, we obtain finer information by factoring the polynomial $Q_{\diamond,\psi}(T)$ in Theorem 1 further, giving a complete hypergeometric decomposition. Our result is as follows.

Corollary 1. *The polynomials $Q_{\diamond,\psi,q}(T)$ factor over $\mathbb{Z}[T]$ according to the following table.*

Family	Factorization	Hypothesis	r
F_4	$(\deg 1)^{12}(\deg 2)^3$	$q \equiv 1 \bmod 4$	2
$\mathsf{F}_2\mathsf{L}_2$	$(1-qT)^6(\deg 1)^2(\deg 2)^5$	$q \equiv 1 \bmod 8$	4
$\mathsf{F}_1\mathsf{L}_3$	$(\deg 6)^3$	$q \equiv 1 \bmod 28$	12
$\mathsf{L}_2\mathsf{L}_2$	$(1-qT)^8(\deg 2)^1(\deg 4)^2$	$q \equiv 1 \bmod 4$	2
L_4	$(1-qT)^2(\deg 4)^4$	$q \equiv 1 \bmod 20$	10

(1)

In Table 1, there may be further factorization depending on ψ and q, and some of these factors may agree. The integer r in Table 1 is such that for $q = p^r$, we have $Q_{\diamond,\psi,q}(T) = (1-qT)^{18}$ under the hypotheses of Theorem A: in other words, if we factor $Q_{\diamond,\psi,q}(T)$ as a product of cyclotomic polynomials ϕ_{m_i}, then $\operatorname{lcm}(m_i) \mid r$.

The case of the Dwork pencil F_4 is due to Dwork [15, §6j, p. 73], and in this case we know that the degree 2 factor occurs with multiplicity 3 and the linear factor occurs with multiplicity 12, as the notation indicates. The factorization in Corollary 1 is motivated by similar work due to Candelas–de la Ossa–Rodriguez-Villegas [6, 7]. (Kloosterman [26] has shown that one can use a group action to describe the distinct factors of the zeta function for any one-parameter monomial deformation of a diagonal hypersurface in weighted projective space; only one of our families, the Dwork pencil, fits within the scope of this work.)

Acknowledgements The authors heartily thank Xenia de la Ossa for her input and many discussions about this project. They thank Simon Judes for sharing his expertise, Frits Beukers for numerous helpful discussions, and Edgar Costa for sharing his code for computing zeta functions. The authors would like to thank the American Institute of Mathematics and its SQuaRE program, the Banff International Research Station, SageMath, and the MATRIX Institute for facilitating their work together. Kelly acknowledges that this material is based upon work supported by the NSF under Award No. DMS-1401446 and the EPSRC under EP/N004922/1. Voight was supported by an NSF CAREER Award (DMS-1151047).

References

1. A. Adolphson and S. Sperber, "Exponential sums and Newton polyhedra: cohomology and estimates" Ann. of Math. (2), 130, 367–406, 1989.
2. M. Artebani, S. Boissière, and A. Sarti "The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces", to appear in Journal des Mathématiques Pures et Appliquées.
3. V.V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties", *J. Algebraic Geom.* 3, 1994, 3, 493–535.
4. P. Berglund and T. Hübsch, "A generalized construction of mirror manifolds", *Nuclear Phys. B*, 393, 1993, 1-2, 377–391.
5. P. Candelas and X. de la Ossa, "The Zeta-Function of a p-adic Manifold, Dwork Theory for Physicists", arxiv:0705.2056v1, 2008.
6. P. Candelas, X. de la Ossa, and F. Rodriguez Villegas, "Calabi–Yau manifolds over finite fields, I", arXiv:hep-th/0012233v1, 2000.
7. P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, "Calabi–Yau manifolds over finite fields II", in *Calabi–Yau varieties and mirror symmetry*, Toronto 2001, 121-157, hep-th/0402133.
8. A. Chiodo and Y. Ruan, "LG/CY correspondence: the state space isomorphism" Adv. Math., 227, Issue 6 (2011), 2157-2188.
9. P. Deligne, "La conjecture de Weil. I", *Inst. Hautes Études Sci. Publ. Math.*, 43,1974,273–307.
10. J. Delsarte, "Nombre de solutions des équations polynomiales sur un corps fini", *Sém. Bourbaki* 39-01, 1951.
11. C. F. Doran, B. Greene, S. Judes, "Families of Quintic Calabi–Yau 3-folds with Discrete Symmetries" Comm Math Phys 280 (2008) pp. 675-725.
12. C.F. Doran, T. Kelly, A. Salerno, S. Sperber, J. Voight, U. Whitcher, "Zeta functions of alternate mirror Calabi–Yau families", arXiv:1612.09249.
13. C.F. Doran, T. Kelly, A. Salerno, S. Sperber, J. Voight, U. Whitcher,, "Arithmetic mirror symmetry for symmetric K3 quartic pencils", in preparation.
14. B. Dwork, "On the rationality of the zeta function of an algebraic variety," *American Journal of Mathematics*, Vol. 82, No. 3, 1960.
15. B. Dwork, "p-adic cycles," *Inst. Hautes Études Sci. Publ. Math.*, 37, 1969, 27–115.
16. L. Fu and D. Wan, "Mirror congruence for rational points on Calabi-Yau varieties", *Asian J. Math.* 10, 2006, 1, 1–10.
17. E. Furtado Gomida, "On the theorem of Artin-Weil", *Soc. Mat. São Paulo* 4, 267–277, 1951.
18. S. Gährs, *Picard–Fuchs equations of special one-parameter families of invertible polynomials*, Ph.D. thesis, Gottfried Wilhelm Leibniz Univ. Hannover, arXiv:1109.3462.
19. S. Gährs, "Picard–Fuchs equations of special one-parameter families of invertible polynomials" *Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds*, Fields Inst. Commun. 67, 285–310,Springer, New York, 2013.
20. B.R. Greene and M. Plesser, "Duality in Calabi-Yau moduli space", *Nuclear Phys. B* 338, 1990, 1, 15–37.

21. B.R. Greene, M. Plesser, and S.-S. Roan, "New constructions of mirror manifolds: probing moduli space far from Fermat points", *Essays on mirror manifolds*, Int. Press, Hong Kong, 1992.
22. J. Greene, "Hypergeometric Functions over Finite Fields", *Trans. of the Am. Math. Soc.*, **301** (1987), no. 1, 77–101.
23. A. Grothendieck, "The cohomology theory of abstract algebraic varieties," *Proc. Internat. Congress Math. (Edinburgh, 1958)*, 103–118, Cambridge Univ. Press, New York, 1960.
24. K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Second Edition, Springer, GTM 84, 1990.
25. N. Katz, *Exponential Sums and Differential Equations*, Annals of Mathematics Studies no. 124, Princeton University Press, 1990.
26. R. Kloosterman, "The zeta function of monomial deformations of Fermat hypersurfaces," *Algebra & Number Theory*, Vol. 1, no. 4, 2007.
27. N. Koblitz, "The number of points on certain families of hyper surfaces over finite fields", *Comp. Math.* Vol. 48, Fasc. 1, 3–23, 1983.
28. M. Krawitz, "FJRW rings and Landau-Ginzburg Mirror Symmetry", arxiv:0906.0796, 2009.
29. M. Kreuzer and H. Skarke, "No mirror symmetry in Landau-Ginzburg spectra!", *Nuclear Phys. B* 388, 1992, no. 1, 113–130.
30. B. Mazur, "Frobenius and the Hodge filtration", *Ann. of Math. (2)*, 98, 1973, 58–95.
31. D. McCarthy, "Transformations of well-poised hypergeometric functions over finite fields", *Finite Fields and their Applications*, **18** (2012), no. 6, 1133–1147.
32. Nikulin, V. Finite automorphism groups of Kähler K3 surfaces. *Transactions of the Moscow Mathematical Society* 38, 1980.
33. Tetsuji Shioda, *An explicit algorithm for computing the Picard number of certain algebraic surfaces*, Amer. J. Math. **108** (1986), no. 2, 415–432.
34. D. Wan, "Mirror symmetry for zeta functions", *Mirror symmetry. V*, AMS/IP Stud. Adv. Math., 38, 2006.