# ON RATIONAL BIANCHI NEWFORMS AND ABELIAN SURFACES WITH QUATERNIONIC MULTIPLICATION

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ABSTRACT. We study the rational Bianchi newforms (weight 2, trivial character, with rational Hecke eigenvalues) in the LMFDB that are not associated to elliptic curves, but instead to abelian surfaces with quaternionic multiplication. Two of these examples exhibit a rather special kind of behaviour: we show they arise from twisted base change of a classical newform with nebentypus character of order 4 and eight inner twists.

## 1. INTRODUCTION

Let $f$ be a classical newform with weight 2 and level $N$. When $f$ has rational coefficients, the Eichler–Shimura construction attaches to $f$ an elliptic curve $E_f$ defined over $\mathbb{Q}$, a quotient of the Jacobian of the modular curve $X_0(N)$, such that $L(f, s) = L(E_f, s)$. More generally, if $f$ has coefficients in a number field $M_f$ of degree $d$, this construction associates a simple abelian variety $A_f$ of dimension $d$ defined over $\mathbb{Q}$, now a quotient of the Jacobian of the modular curve $X_1(N)$, such that $\text{End}(A_f) \otimes \mathbb{Q} \simeq M_f$ and $L(A_f, s) = \prod_\sigma L(\sigma(f), s)$, the product over all embeddings $\sigma \colon M_f \hookrightarrow \mathbb{C}$. (It may happen that $A_f$ acquires additional endomorphisms over a number field, and in particular $A_f$ may not be geometrically simple.)

It is natural to seek such a relationship for modular forms over other number fields. In the case of totally real number fields, in many instances (for example, over fields of odd degree) a construction analogous to the Eichler–Shimura construction is obtained by replacing the modular curve by a suitable Shimura curve. However, a general such construction over number fields is not known. So with the simplest case in mind, we are led to ask: given a Bianchi newform $F$ of weight 2 over an imaginary quadratic field $K$, is there an abelian variety $A_F$ defined over $K$ whose $L$-series matches that of $F$ (as above)?

A precise answer to this question, consistent with the predictions of the Langlands philosophy, is provided affirmatively by a conjecture of Taylor [Tay95, Conjecture 3]—but with a wrinkle. Let $M_F$ be the number field generated by the Hecke eigenvalues of $F$, and let $d = [M_F : \mathbb{Q}]$. Then conjecturally there is an abelian variety $A_F$ of dimension $2d$ defined over $K$ with *quaternionic multiplication* (QM) by a quaternion algebra $B_F$ over $M_F$ (and defined over $K$) such that

$$L(A_F, s) = \prod_{\sigma : M_F \hookrightarrow \mathbb{C}} L(\sigma(F), s)^2. \tag{1.1}$$

In this formulation, it may happen that the quaternion algebra $B_F \simeq \text{M}_2(M_F)$ is split, in which case $A_F \sim (A'_F)^2$ is isogenous over $K$ to the square of an abelian variety $A'_F$ of

dimension $d$ and $L(A'_F, s) = \prod_\sigma L(\sigma(F), s)$. (In general, if the base field $K$ has a real place, then $B_F$ is necessarily split, which explains the simplifications above.)

In particular, if $F$ has rational coefficients (i.e., $M_F = \mathbb{Q}$), then attached to $F$ is conjecturally a QM abelian surface that is often the square of an elliptic curve. Examples of the phenomenon where the quaternion algebra is a division algebra were exhibited by Cremona [Cre84, Cre92] in the case where $F$ is the base-change of a classical newform with quadratic coefficients, or a quadratic twist of such a form. More recently, Schembri [Sch19] exhibited examples which are not base change. QM abelian surfaces have been called *false generalized elliptic curves* by Taylor [Tay95], *false elliptic curves* by Serre (as attributed by Deligne–Rapoport [DR73, p. DeRa-12]), and *fake elliptic curves* by others. Perhaps this terminology would be more appropriate for the non-existent geometric object whose $L$-function is equal to $L(F, s)$.

Given that the situation is rather unusual, in this paper we study it explicitly. We consider the Bianchi newforms $F$ of weight 2 with rational coefficients in the *L-Functions and Modular Forms Database* (LMFDB) [LMF20]. For thousands of such forms, we can explicitly attach an elliptic curve $E_F$ such that $L(F, s) = L(E_F, s)$, with $E_F$ unique up to isogeny over $K$. For the remaining forms, we can find no such elliptic curve—in Section 2, we list these forms and summarize what is known about them. In all these remaining examples, we can attach a QM abelian surface $A$ over $K$ such that $L(A, s) = L(F, s)^2$, in agreement with the above conjecture.

Two examples in the list stand out, and because of their exotic properties they command the majority of our attention here. We consider the rational Bianchi newform $F$ of weight 2 with LMFDB label[1] 2.0.7.1-30625.1-c over $K = \mathbb{Q}(\sqrt{-7})$, along with a certain quadratic twist $G$, which has label 2.0.7.1-30625.1-e. The newform $F$ has rational coefficients and is a twist of a base-change form, but there is no *quadratic* twist of $F$ which is base-change; in fact the simplest twist which is base-change is by a character of order 8, so has neither trivial nebentypus nor rational coefficients.

Our main result (combining Proposition 6.2, Theorem 6.9, and Proposition 7.5) is the following.

**Theorem A.** *Let $F$ be the rational Bianchi newform over $K := \mathbb{Q}(\sqrt{-7})$ of weight 2, level $(175)$, and trivial character with LMFDB label 2.0.7.1-30625.1-c. Then the following statements hold.*

(a) *There exists a classical modular form $f$ of weight 2, level 1225, character of conductor 35 and order 4, and Hecke field $\mathbb{Q}(\zeta_{24})$, such that*
$$F = f_K \otimes \psi,$$
*where $f_K$ is the base change of $f$ to $K$ and $\psi$ is a character of conductor $(5\sqrt{-7})$ and order 8.*

(b) *There exists an explicit genus 2 curve $X_F$ over $K$ whose Jacobian $J_F := \mathrm{Jac}(X_F)$ has QM by a quaternion algebra of discriminant 15 and*
$$L_{\mathfrak{p}}(X_F, T) = L_{\mathfrak{p}}(J_F, T) = L_{\mathfrak{p}}(F, T)^2$$

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[1]Here, 2.0.7.1 is the LMFDB label for the base field $K = \mathbb{Q}(\sqrt{-7})$ and 30625.1 the label for the level ideal $(175)$, which has norm 30625. The final c is the alphabetic label for this specific newform at that level. We use either full labels such as 2.0.7.1-30625.1-c for Bianchi newforms, or the shorter version 30625.1-c which omits the field when that is clear from the context.

for all primes $\mathfrak{p} \neq (2), (5), (\sqrt{-7})$. *Moreover, all endomorphisms of* $J_F$ *are defined over* $K$.

(c) *Let* $A_f$ *be the abelian eightfold over* $\mathbb{Q}$ *attached to* $f$. *Then* $\mathrm{End}(A_f) = \mathbb{Z}[\zeta_{24}]$. *Let* $A' := (A_f)_K^{(\psi)}$ *be the base change of* $A_f$ *to* $K$ *twisted by* $\psi$ *(relative to an automorphism of* $A_f$ *of order 8). Then there is an isogeny* $A' \sim J_F^2 \times J_G^2$ *over* $K$, *where* $G = F \otimes \psi^4$ *is a quadratic twist of* $F$ *(with LMFDB label* 2.0.7.1-30625.1-e*), and* $J_G$ *the corresponding quadratic twist of* $J_F$.

The form is described in detail in Section 4, and the twists are given explicitly in Section 6. An equation for the curve $X_F$ in Theorem 1(b) is given in (7.4).

**Contents.** The paper is organized as follows. In Section 2 we survey Bianchi newforms in the LMFDB. After quickly setting up notation for characters in Section 3, we describe the classical modular form $f$ in Section 4; then we consider its base change and twists in Sections 5–6. We then give a model for the curve $X_F$ in Section 7 and in Section 8 briefly describe related Bianchi modular forms in the same level. We conclude in Section 9 with some connections to the Paramodular Conjecture.

## 2. Rational Bianchi newforms with no associated elliptic curve

The LMFDB [LMF20] currently contains rational Bianchi newforms over each of the five fields $K = \mathbb{Q}(\sqrt{d})$ for $d = -3, -4, -7, -8, -11$, as computed by Cremona [Cre84], of all levels of norm up to a bound depending on the field, currently (August 2020): $150\,000$ for $d = -3$, $100\,000$ for $d = -4$, and $50\,000$ for $d = -7, -8, -11$. See Table 2.1. The total number of rational newforms within these ranges is $185\,581$, of which $5\,917$ arise from base change from $\mathbb{Q}$. All but 40 of the $185\,581$ newforms $F$ have a candidate associated elliptic curve $E$ defined over the field $K$ in the sense given in the introduction, namely $L(E, s) = L(F, s)$; or equivalently that they have the same compatible system of $\ell$-adic Galois representations. It is possible (in principle) to prove modularity of all these elliptic curves, for example using the Faltings–Serre–Livné method (as detailed by Dieulefait–Guerberoff–Pacetti [DGP10]). This is being carried out systematically by Mattia Sanna (Warwick) for his PhD thesis [San20], using a combination of 2-adic and 3-adic methods combined with recent modularity lifting theorems.

Of the 40 remaining rational Bianchi newforms without associated elliptic curves, six arise from base change of classical modular forms over $\mathbb{Q}$ with trivial character, real quadratic coefficients and inner twists: see [Cre92]. A further 18 are twisted base change, and for 16 of these 18, the base-change form itself and the form over $\mathbb{Q}$ have trivial character; the other two will be studied in this paper. The remaining 16 forms are associated to abelian surfaces

over the base field with quaternionic multiplication (QM), and were considered by Schembri [Sch19].

| Field discriminant | $-4$ | $-8$ | $-3$ | $-7$ | $-11$ | Total |
|---|---|---|---|---|---|---|
| Level norm bound | 100000 | 50000 | 150000 | 50000 | 50000 | |
| Rational newform counts | | | | | | |
| All rational newforms | 40030 | 36843 | 42343 | 35824 | 30541 | 185581 |
| Not twist of base change | 38844 | 35536 | 40986 | 34830 | 29468 | 179664 |
| Base change of rational newform | 908 | 809 | 955 | 518 | 504 | 3694 |
| Base change of quadratic newform | 16 | 10 | 28 | 24 | 13 | 91 |
| Twist of base change of rational newform | 240 | 350 | 328 | 378 | 492 | 1788 |
| Twist of base change of quadratic newform | 22 | 138 | 46 | 72 | 64 | 342 |
| Twist of base change of higher-dimensional newform | 0 | 0 | 0 | 2 | 0 | 2 |
| Rational newforms with no elliptic curve | | | | | | |
| Total | 4 | 8 | 21 | 7 | 0 | 40 |
| Quadratic twist of base change | 0 | 8 | 9 | 5 | 0 | 22 |
| Higher order twist of base change | 0 | 0 | 0 | 2 | 0 | 2 |
| Not twist of base change | 4 | 0 | 12 | 0 | 0 | 16 |

Table 2.1: Summary of Bianchi modular form data in the LMFDB

There is a natural action of $\mathrm{Gal}(K \,|\, \mathbb{Q}) = \langle \tau \rangle$ on rational Bianchi newforms, $f \mapsto \tau(f)$, where $a_{\mathfrak{p}}(\tau(f)) = a_{\tau(\mathfrak{p})}(f)$; if $g = \tau(f)$ then we say that $f, g$ are *Galois conjugate*. (Our forms have rational coefficients, so there should be no confusion with other Galois actions.)

Organizing according to field and level, we have the following further details about the 40 rational newforms with no associated elliptic curve. These are either base-change, or come in Galois conjugate pairs. In some cases the forms mentioned are twists of base-change, but the base-change newform itself has too large a level to be in the current database, so does not yet have a label.

- $K = \mathbb{Q}(\sqrt{-1})$, LMFDB label 2.0.4.1: four cases, none of which are base-change (even up to twist).
  - 34225.3-a is not a base-change or twist of base-change. It is associated to a simple abelian surface with QM over $K$, the Jacobian of the genus 2 curve $C_1$ found by Schembri [Sch19, Theorem 1.1].
  - 34225.3-b is a quadratic twist of 34225.3-a.
  - 34225.7-a and 34225.7-b are Galois conjugates of the previous two.
- $K = \mathbb{Q}(\sqrt{-2})$, LMFDB label 2.0.8.1: eight cases, all base-change (up to twist).
  - 5625.1-b and its Galois conjugate 5625.3-b are both twists of base-changes of classical newforms in $S_2(28800)$.
  - 6561.5-a and its Galois conjugate 6561.5-d are both twists of a Bianchi newform at level $(2592)$, which is a base-change from $S_2(20736)$. The classical newform

has coefficients in $\mathbb{Q}(\sqrt{15})$ and attached to the Bianchi newform is an abelian surface with QM of discriminant 10, but an explicit equation is not known.

– 21609.3-b and 21609.3-c are both twists of a Bianchi newform at level (14112), which is a base-change from $S_2(11289)$. The classical newform has coefficients in $\mathbb{Q}(\sqrt{7})$ and attached to the Bianchi newform is an abelian surface with QM of discriminant 14. Again, an explicit equation is not known.

– 21609.1-b and 21609.1-c are Galois conjugates of the previous two.

• $K = \mathbb{Q}(\sqrt{-3})$, LMFDB label 2.0.3.1: 21 cases, of which 12 are not twists of base-change but have associated genus 2 curves with QM:

– 61009.1-a is associated to the genus 2 curve $C_2$ found by Schembri [Sch19, Theorem 1.1], while 61009.1-b is a quadratic twist of this, and 61009.9-a and 61009.9-b are their Galois conjugates.

– 67081.3-a is associated to the genus 2 curve $C_3$ found by Schembri [Sch19, Theorem 1.1], while 67081.3-b is a quadratic twist of this, and 67081.7-a and 67081.7-b are their Galois conjugates.

– 123201.1-b is associated to the genus 2 curve $C_4$ found by Schembri [Sch19, Theorem 1.1], while 123201.1-c is a quadratic twist of this, and 123201.3-b and 123201.3-c are their Galois conjugates.

The other nine cases are twists of base-changes of forms with trivial character, coefficients in a real quadratic field and an inner twist; in these cases, explicit equations are not known:

– 5625.1-a is the base-change of 225.2.a.f, which has CM by $-15$, coefficients in $\mathbb{Q}(\sqrt{5})$, and an inner twist. This is an example of the phenomenon first described by Cremona [Cre92].

– 6561.1-b is base-change of 243.2.a.d, with coefficients in $\mathbb{Q}(\sqrt{6})$ and inner twist, attached to an abelian surface with QM of discriminant 6.

– 30625.1-a and its Galois conjugate 30625.3-a are twists of base-change of forms in $S_2(11025)$ with CM by $-15$.

– 50625.1-c and its Galois conjugate 50625.1-d are base-changes of 675.2.a.l and 675.2.a.m respectively, with coefficients in $\mathbb{Q}(\sqrt{2})$ and inner twist, attached to abelian surfaces with QM of discriminant 6.

– 65536.1-b and its Galois conjugate 65536.1-e are twists of a form at level (768), which is base-change of 2304.2.a.p, with CM by $-6$, coefficients in $\mathbb{Q}(\sqrt{2})$ and inner twist.

– 104976.1-a is base-change of 972.2.a.e, with coefficients in $\mathbb{Q}(\sqrt{2})$ and inner twist, attached to an abelian surface with QM of discriminant 6.

• $K = \mathbb{Q}(\sqrt{-7})$, LMFDB label 2.0.7.1: seven cases. Of these, two are the forms $F$ and $G$ in Theorem 1 and discussed in detail in Section 8 below. In addition, we have

– 30625.1-d: see Section 8.

– 40000.1-b and its Galois conjugate 40000.7-b are twists of 30625.1-d.

– 10000.1-b and its Galois conjugate 10000.5-b are twists of forms which are the base-change of forms in $S_2(19600)$ with coefficients in $\mathbb{Q}(\sqrt{5})$ and inner twist.

• $K = \mathbb{Q}(\sqrt{-11})$, LMFDB label 2.0.11.1: none.

## 3. CHARACTERS

We begin by setting some notation, in particular we define some characters. Let $\mathbb{Q}^{\mathrm{al}}$ be an algebraic closure of $\mathbb{Q}$. By a *character* over $\mathbb{Q}$ we will mean either a Dirichlet character or the character of the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q})$ attached to it via class field theory; we define the same over a number field $K$ allowing Hecke characters (of finite order). If $\chi$ is a character over $\mathbb{Q}$, we may talk of its *restriction* $\chi|_K$ to $K$ thinking of it as a character of $\mathrm{Gal}_{\mathbb{Q}}$ and restricted to $\mathrm{Gal}_K := \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, K)$.

Let $N := 1225 = 5^2 7^2$; we now define some Dirichlet characters of modulus $N$. Denote by $\chi_5$ the mod 5 cyclotomic character of order 4 (with LMFDB label $\chi_{1225}(393, \cdot)$) and by $\chi_{-7}$ the quadratic character of conductor 7 (label $\chi_{1225}(1126, \cdot)$). As characters of $\mathrm{Gal}_{\mathbb{Q}}$, these cut out the cyclic extensions $\mathbb{Q}(\zeta_5)$ and

$$K := \mathbb{Q}(\sqrt{-7}) \tag{3.1}$$

respectively. Define $\varepsilon := \chi_5 \chi_{-7}$ (label $\chi_{1225}(293, \cdot)$); then $\varepsilon$ has conductor 35 and order 4. Let

$$H := \langle \chi_5 \rangle \times \langle \chi_{-7} \rangle \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \tag{3.2}$$

be the group generated by $\chi_{-7}$ and $\chi_5$.

After restricting to $K$, the character $\chi_{-7}|_K$ is trivial, so $\varepsilon|_K = \chi_5|_K$. Also, while $\chi_5$ is not a square (since $\mathbb{Q}$ has no cyclic extension of degree 8 ramified only at 5), its restriction to $K$ is a square, since 5 is inert in $K$. We denote by $\psi$ one of the two order eight characters of $K$ such that $\psi^{-2} = \chi_5|_K = \varepsilon|_K$; the other such character is $\psi^5$. The conductor of $\psi$ is $(5\sqrt{-7})$.

The reason why the character $\psi$ must ramify at the prime $\sqrt{-7}$ can be explained as follows. The multiplicative group of the residue field of the prime ideal $(5)$ is $\mathbb{F}_{25}^{\times}$ and so admits an order 8 character $\psi_5$. Such a character is odd, i.e., $\psi_5(-1) = -1$. A finite order Hecke character has infinity type $(0, 0)$ (i.e. it is trivial at the Archimedean places), hence if it is unramified outside 5, such a character would not be trivial at $-1$. However, the quadratic character $\psi_{\sqrt{-7}}$ is also odd, and the character $\psi$ corresponds to the Hecke character ramified at $\{5, \sqrt{-7}\}$ with such ramification at both primes.

## 4. A CLASSICAL NEWFORM OF LEVEL 1225 AND AN ABELIAN EIGHTFOLD

In this section, we investigate a certain special classical newform; all statements below can be confirmed using MAGMA [BCP97] or PARI/GP [PAR21].

We retain notation from the previous section. Consider the newspace $S_2(N, \varepsilon)^{\mathrm{new}}$. Its relative dimension as a vector space over $\mathbb{Q}(\varepsilon) \simeq \mathbb{Q}(i)$ is 56, and it decomposes under the action of the Hecke algebra into 7 irreducible components, of relative dimensions $2, 2, 4, 8, 12, 12, 16$. Their absolute dimensions, as $\mathbb{Q}$-vector spaces, are obtained by multiplying by $[\mathbb{Q}(\varepsilon) : \mathbb{Q}] = 2$.

Let $f$ be any one of the four Galois conjugate newforms in the unique 4-dimensional component; these four form a single orbit under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q}(\varepsilon))$. Of the eight $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q})$-conjugates of $f$, four are in $S_2(N, \varepsilon)$ and the other four in $S_2(N, \overline{\varepsilon})$, where $\overline{\varepsilon} = \varepsilon^3 = \overline{\chi_5} \chi_{-7}$. The form $f$ does not have CM and its Hecke field is $M_f := \mathbb{Q}(\zeta_{24})$.

*Remark* 4.1. The form $f$ has $Nk^2 = 4900 \geq 4000$, and so is just outside the current range of forms in the LMFDB.

**Proposition 4.2.** *For each character $\chi \in H$, the twist $f \otimes \chi$ is a conjugate of $f$, i.e., there exists $\sigma \in \mathrm{Gal}(M_f \,|\, \mathbb{Q})$ such that $f \otimes \chi = \sigma(f)$.*

*Proof.* First observe that the level $N$ is divisible by the square of the conductor of $\varepsilon$, and also that neither $\chi_{-7}$ nor $\chi_5$ is a square. This implies that the twists have the same level as $N$ [Shi94, Proposition 3.64].

Let $\chi \in H$. Then the nebentypus character of $f \otimes \chi$ is $\varepsilon\chi^2$, which equals either $\varepsilon$ or $\bar{\varepsilon}$. Hence the twists are newforms in either $S_2(N, \varepsilon)^{\mathrm{new}}$ or $S_2(N, \bar{\varepsilon})^{\mathrm{new}}$. The $\mathrm{Gal}_{\mathbb{Q}}$-orbit of $f \otimes \chi$ has size at most 8 (but might *a priori* be smaller). However, $a_2(f) = -\zeta_{24}^3$ is an eighth root of unity, while the $a_2$ coefficients of all the newforms of smaller dimension in $S_2(N, \varepsilon)$ (which lie in $\mathbb{Q}(\zeta_{12})$) are not roots of unity, so cannot be equal to $\chi(2)a_p(f)$. Hence $f \otimes \chi$ has the same Hecke field as $f$ and by the uniqueness of the $\mathbb{Q}$-dimension 8 component of $S_2(N, \varepsilon)^{\mathrm{new}}$ must be conjugates of $f$. $\qquad\square$

Proposition 4.2 says that the newform $f$ has eight *inner twists* $(\sigma, \chi)$ (including the trivial twist, where both $\chi$ and $\sigma$ are trivial). To keep track of these inner twists, we use the following notation: let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{24}) \,|\, \mathbb{Q})$ be the automorphism such that $\sigma_a(\zeta_{24}) = \zeta_{24}^a$. Checking the first few Fourier coefficients $a_p(f)$ we see that the inner twists are defined by the data $(\sigma, \chi)$ in Table 4.3.

| $\chi$ | $1$ | $\chi_5$ | $\chi_5^2$ | $\chi_5^3$ | $\chi_{-7}$ | $\chi_5\chi_{-7}$ | $\chi_5^2\chi_{-7}$ | $\chi_5^3\chi_{-7}$ |
|---|---|---|---|---|---|---|---|---|
| $\sigma$ | $\sigma_1$ | $\sigma_{19}$ | $\sigma_{13}$ | $\sigma_7$ | $\sigma_{17}$ | $\sigma_{11}$ | $\sigma_5$ | $\sigma_{23}$ |

Table 4.3: Inner twists of $f$

By the Eichler–Shimura construction, attached to (the Galois orbit of) $f$ is a simple abelian 8-fold $A = A_f$, defined over $\mathbb{Q}$ with $\mathrm{End}(A)_{\mathbb{Q}} := \mathrm{End}(A) \otimes \mathbb{Q} \simeq M_f$. Write $A_K := A \times_{\mathbb{Q}} K$ for the base extension of $A$ to $K = \mathbb{Q}(\sqrt{-7})$, and $A^{\mathrm{al}} := A \times_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}}$ for the base extension of $A$ to $\mathbb{Q}^{\mathrm{al}}$. To determine how $A$ splits over $\mathbb{Q}^{\mathrm{al}}$ we must determine the *geometric* endomorphism algebra $\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}})_{\mathbb{Q}}$, which is generated over $\mathbb{Q}(\zeta_{24})$ by endomorphisms arising from inner twists of $f$, by Ribet–Pyle [Rib80, Rib04, Pyl04] theory. We use the implementation in MAGMA [BCP97] due to Quer [Que09] and Stein.

**Proposition 4.4.** *The following statements hold.*

(a) *Each inner twist $(\sigma, \chi)$ of $f$ gives rise to an endomorphism $\xi_{(\sigma,\chi)}$ of $A^{\mathrm{al}}$, defined over the field $\mathbb{Q}(\chi)$ cut out by $\chi$ and satisfying*

$$\alpha\xi_{(\sigma,\chi)} = \xi_{(\sigma,\chi)}\sigma(\alpha) \tag{4.5}$$

*for all $\alpha \in \mathbb{Q}(\zeta_{24})$. The endomorphisms $\xi_{(\sigma,\chi)}$ are a $\mathbb{Q}(\zeta_{24})$-basis for $\mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}}$.*

(b) *We have $\mathrm{End}(A_K)_{\mathbb{Q}} \simeq M_2(\mathbb{Q}(\zeta_8))$, with centre $Z(\mathrm{End}(A_K)) = \mathbb{Z}[\zeta_8]$.*

(c) *We have $\mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}} \simeq M_4(B)$ where $B \simeq \left(\dfrac{3,5}{\mathbb{Q}}\right)$ is the division quaternion algebra of discriminant 15 over $\mathbb{Q}$. All endomorphisms of $A^{\mathrm{al}}$ are defined over $K' := \mathbb{Q}(\zeta_5, \sqrt{-7})$.*

(d) *There exists a simple abelian surface $S$ defined over $K$ with $\mathrm{End}(S)_{\mathbb{Q}} \simeq B$ and an isogeny $A \sim S^4$ defined over $K'$.*

*Proof.* Statement (a) is an application of work of Ribet [Rib80, (5.5)] together with the computation of inner twists above (Proposition 4.2). Statement (b) follows from (a): over $K = \mathbb{Q}(\sqrt{-7})$, the abelian variety $A_K$ obtains the endomorphism $\xi_{(\sigma_{17},\chi_{-7})}$; since the fixed

field of $\sigma_{17}$ is $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\zeta_8)(\sqrt{-3})$, we conclude that

$$\text{End}(A_K)_{\mathbb{Q}} \simeq \left(\frac{-3, -7}{\mathbb{Q}(\zeta_8)}\right) \simeq \text{M}_2(\mathbb{Q}(\zeta_8)). \tag{4.6}$$

We have $\zeta_8 \in \text{End}(A_K)$ because already $\mathbb{Z}[\zeta_{24}] \subseteq \text{End}(A)$, and $\mathbb{Q}(\zeta_8) \subseteq Z(\text{End}(A_K)_{\mathbb{Q}})$.

We finish (c) and (d) following Pyle [Pyl04]. Over $\mathbb{Q}^{\text{al}}$, we have $A \sim S^r$ where $S$ (over $\mathbb{Q}^{\text{al}}$) is simple, called a *building block*. We have $\dim S = 1$ or $2$, and these cases are distinguished by a Brauer class; MAGMA confirms that $\dim S = 2$, $r = 4$, and therefore $\text{End}(S)_{\mathbb{Q}} \simeq B$. Moreover, calculating the obstruction to descent [Pyl04, Proposition 5.2], we find that $S$ and its endomorphism algebra may be defined, up to isogeny, over $K$. $\qquad\square$

The inner twist relation $f \otimes \chi = \sigma(f)$ implies, on comparing the nebentypus character on both sides, that

$$\varepsilon \chi^2 = \sigma_\chi(\varepsilon) \tag{4.7}$$

and that for all but finitely many $p$, we have

$$a_p(f)\chi(p) = \sigma(a_p(f)). \tag{4.8}$$

Together these imply that $a_p(f)^2/\varepsilon(p)$ is fixed by all $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q})$, so we conclude that $a_p(f)^2/\varepsilon(p) \in \mathbb{Q}$ for all but finitely many $p$ (and this relation holds in generality).

The $L$-function of $A$ (over $\mathbb{Q}$) is

$$L(A, s) = \prod_{\chi \in H} L(f \otimes \chi, s), \tag{4.9}$$

since the 8 Galois conjugates of $f$ are precisely the 8 twists $f \otimes \chi$ for $\chi \in H$. After base change to $K' = \mathbb{Q}(\zeta_5, \sqrt{-7})$ to trivialize all the characters, by Proposition 4.4(d) we find that

$$L(A, K', s) = L(S, K', s)^4 = L(f_{K'}, s)^8 \tag{4.10}$$

so that

$$L(S, K', s) = L(f_{K'}, s)^2. \tag{4.11}$$

## 5. Base change to $\mathbb{Q}(\sqrt{-7})$

In light of the results of the previous section, we now move to $K = \mathbb{Q}(\sqrt{-7})$. By Proposition 4.4(b), we have

$$A_K \sim C_0^2, \tag{5.1}$$

where $C_0$ is a simple abelian fourfold defined over $K$ with $\text{End}(C_0)_{\mathbb{Q}} \simeq \mathbb{Q}(\zeta_8)$.

We now consider the base-change $f_K$ of the newform $f$ to $K$, a Bianchi modular form of weight 2. In the classical language of automorphic forms, the base change is known as the Doi–Naganuma lift [DN70] and can be obtained using the theta function method as generalized to arbitrary weight, level, and character by Friedberg [Fri83, Theorems 3.1, 3.2]. More generally, base change can be understood in the language of automorphic representations: see Gerardin–Labesse [GL79] or Langlands [Lan80].

Although we do not use it explicitly, we recall the formula for the Hecke eigenvalues of $f_K$. If $p$ splits in $K$ and $\mathfrak{p}$ lies over $p$ then $a_{\mathfrak{p}}(f_K) = a_p(f)$; if $p$ is inert in $K$ then $a_{(p)}(f_K) = a_p(f)^2 - 2\varepsilon(p)p$.

Recall the character $\psi$ of order 8 defined in Section 3.

**Proposition 5.2.** *The base-change $f_K$ has the following properties.*

(a) *We have $f_K \in S_2(\mathfrak{N}, \psi^{-2})$, so $f_K$ has level $\mathfrak{N} = (175) = (5)^2(\sqrt{-7})^2$ and nebentypus character $\varepsilon|_K = \psi^{-2}$.*

(b) *The Fourier coefficients of $f_K$ lie in $\mathbb{Q}(\zeta_8)$.*

(c) *For each inner twist $(\sigma, \chi)$ of $f$, the base change $f_K$ has inner twist by $(\sigma, \chi|_K)$; so $f_K$ has four distinct inner twists (including the trivial inner twist).*

*Proof.* The level of $f_K$ is determined uniquely by the properties that it is stable under Galois conjugation and of norm $N^2/\operatorname{Disc}(K)^2 = 1225^2/7^2 = 30625$ [Ţur18, Lemma 5.2], where we use the fact that the $-7$-twist of $f$ has the same level by Proposition 4.2: $f$ has inner twist by $(\sigma_{17}, \chi_{-7})$. Alternatively, since $1225 = 5^2 7^2$ the level of $f_K$ divides $(5)^2(\sqrt{-7})^2$ but this is already equal to the conductor of the character $\varepsilon|_K = \psi^{-2}$ of $f_K$. (This also follows from the recognition of $f_K \otimes \psi \in S_2(\Gamma_0(\mathfrak{N}))$ in Proposition 6.2 below.)

Let $(\sigma, \chi)$ be any of the inner twists of $f$, so $\sigma(f) = f \otimes \chi$. Restricting to $K$ gives $\sigma(f_K) = f_K \otimes \chi|_K$. Taking the inner twist to be $(\sigma_{17}, \chi_{-7})$ shows that $\sigma_{17}(f_K) = f_K$, so the coefficients of $f_K$ lie in the fixed field of $\sigma_{17}$, which is $\mathbb{Q}(\zeta_8)$. $\square$

After restriction to $K$, the characters in $H$ coincide in pairs to give all the even powers of $\psi$. Hence we have

$$L(A, K, s) = \prod_{i=1}^{4} L(f_K \otimes \psi^{2i}, s)^2, \tag{5.3}$$

and hence from (5.1) that

$$L(C_0, s) = \prod_{i=1}^{4} L(f_K \otimes \psi^{2i}, s). \tag{5.4}$$

## 6. Twisting

We now investigate twists of the base change form $f_K \in S_2(\mathfrak{N}, \psi^{-2})$ from the previous section. Define

$$F := f_K \otimes \psi \quad \text{and} \quad G := f_K \otimes \psi^5 = F \otimes \psi^4. \tag{6.1}$$

**Proposition 6.2.** *The twists $F, G \in S_2(\mathfrak{N})$ have rational eigenvalues; the form $F$ has LMFDB label 2.0.7.1-30625.1-c, while $G$ has label 2.0.7.1-30625.1-e.*

*Proof.* Note that twisting leaves the level unchanged and the twisted forms have trivial nebentypus.

It is enough to show that $F$ is fixed by all $\sigma \in \operatorname{Gal}(\mathbb{Q}(\zeta_8)\,|\,\mathbb{Q})$, since twisting by the quadratic character $\psi^4$ preserves rationality. Let $(\sigma, \chi)$ be an inner twist of $f$. Then $\sigma(f_K) = f_K \otimes \chi|_K$ and

$$\sigma(f_K \otimes \psi) = \sigma(f_K) \otimes \sigma(\psi) = f_K \otimes \chi|_K \sigma(\psi). \tag{6.3}$$

Since there is an inner twist for every $\sigma \in \operatorname{Gal}(\mathbb{Q}(\zeta_8)\,|\,\mathbb{Q})$, the result is equivalent to establishing the character relations

$$\psi = \chi|_K \sigma(\psi) \tag{6.4}$$

for all such $\chi$. Inspection of Table 4.3 shows that for $a \in (\mathbb{Z}/8\mathbb{Z})^\times$,

$$(\chi_5^{(a-1)/2})|_K = (\psi^{-2})^{(a-1)/2} = \psi/\psi^a = \psi/\sigma_a(\psi) \tag{6.5}$$

verifying (6.4).

Having identified a base change, we now examine the space of Bianchi newforms of level $\mathfrak{N}$, weight 2 and trivial nebentypus character, a space with LMFDB label 2.0.7.1-30625.1. The new subspace has dimension 60 and contains 8 newforms with rational Fourier coefficients. Of these one matches $F$, namely 2.0.7.1-30625.1-c, while $G$ matches 2.0.7.1-30625.1-e, and is the $\sqrt{5}$-twist of $F$ (since $\psi^4 = \chi_5^2|_K$, the quadratic character associated to $K(\sqrt{5}) \supseteq K$).  □

*Remark* 6.6. It is straightforward to identify whether a Bianchi newform $F$ is a twist of base change since if $F$ and $\tau(F)$ are twist-equivalent then either $F$ is a twist of base change or is induced from a Hecke character [LR98]. Furthermore, for two Bianchi newforms $F_1$ and $F_2$ with trivial nebentypus, if $a_{\mathfrak{p}}(F_1)^2 = a_{\mathfrak{p}}(F_2)^2$ for all but finitely many primes $\mathfrak{p}$ then $F_1$ and $F_2$ are twist-equivalent [Ram00].

*Remark* 6.7. In fact, all eight rational Bianchi newforms in the space $S_2(\mathfrak{N})$ (with LMFDB label 2.0.7.1-30625.1) give interesting examples: we briefly describe the other six below (Section 8). The LMFDB does not contain details of Bianchi newforms with nontrivial nebentypus character, such as the form $f_K$ itself.

That the twisted forms $F$ and $G$ have rational eigenvalues means that one can reasonably expect the corresponding twist of the fourfold $C_0$ to split as a product of QM-surfaces. Recall (see after (5.1)) that $\mathrm{End}(C_0)_{\mathbb{Q}} \simeq \mathbb{Q}(\zeta_8)$; we choose an automorphism $\xi \in \mathrm{Aut}(C_0)$ of order 8. With this choice, we may twist $C_0$ via $\psi$ (relative to $\xi$), to obtain a new fourfold defined over $K$, denoted $C$:

$$C := C_0^{(\psi)}. \tag{6.8}$$

**Theorem 6.9.** *There is an isogeny $C \sim S_F \times S_G$ defined over $K$, where $S_F, S_G$ are abelian surfaces defined over $K$ with QM by the rational quaternion algebra $B$ of discriminant 15, satisfying $L(S_F, s) = L(F, s)^2$ and $L(S_G, s) = L(G, s)^2$.*

*Proof.* Let $\ell \equiv 1 \pmod 8$ be prime and let $V_\ell C_0$ be the $\ell$-adic Tate representation of $C_0$, a free $\mathbb{Q}_\ell[\xi]$-module of rank 2 where $\xi^4 = -1$; choosing a primitive eighth root of unity $\zeta_8 \in \mathbb{Q}_\ell$, we may write

$$V_\ell C_0 \simeq \bigoplus_{k \in (\mathbb{Z}/8\mathbb{Z})^\times} W_{\ell,k} \tag{6.10}$$

where each $W_{\ell,k}$ is a $\mathbb{Q}_\ell$-vector space of dimension 2 with $\xi$ acting by the scalar $\zeta_8^k$.

Now $V_\ell C_0$ is also a representation of $\mathrm{Gal}_K$; comparing with the $L$-series decomposition (5.4), as Galois representations we have (up to the choice of $\zeta_8$)

$$W_{\ell,k} \simeq W_{\ell,1} \otimes \psi^{k-1}. \tag{6.11}$$

On the twist, we have the Galois representation

$$V_\ell C = V_\ell C_0 \otimes_{\mathbb{Q}_\ell[\xi]} \mathbb{Q}_\ell[\xi](\psi) \tag{6.12}$$

where $\mathbb{Q}_\ell[\xi](\psi)$ is $\mathbb{Q}_\ell[\xi]$ where $\mathrm{Gal}_K$ acts via $\psi$: explicitly, on the eigenspace of $\mathbb{Q}_\ell[\xi]$ where $\xi$ acts by $\zeta_8^k$, now $\mathrm{Gal}_K$ acts via $\psi^k$. In terms of the splitting (6.10) above, we have

$$V_\ell C \simeq \bigoplus_{k \in (\mathbb{Z}/8\mathbb{Z})^\times} W_{\ell,k} \otimes \psi^k \simeq \bigoplus_k W_{\ell,1} \otimes \psi^{2k-1} \tag{6.13}$$

and thus

$$L(C, K, s) = L(f_K \otimes \psi, s) L(f_K \otimes \psi^5, s) L(f_K \otimes \psi^9, s) L(f_K \otimes \psi^{13}, s)$$
$$= L(F, s)^2 L(G, s)^2. \tag{6.14}$$

Let $L \supseteq K' = K(\zeta_5)$ be the cyclic degree 8 extension of $K$ cut out by $\psi$. By work of Kida [Kid95, p. 53] we have

$$\operatorname{Res}_{L|K} C_{0,L} \sim_K \prod_{i=0}^{7} C_0^{(\psi^i)}; \tag{6.15}$$

i.e., the restriction of scalars of the base change of $A$ to $L$ is isogenous over $K$ to the product of the twists of $A$ by powers of $\psi$. From the $L$-series decomposition (5.4) and the theorem of Faltings, we conclude that

$$\prod_{i=0}^{7} C_0^{(\psi^i)} \sim_K C_0^4 \times C^4. \tag{6.16}$$

Let $S$ be the building block from Proposition 4.4(d); up to isogeny (over $K'$), we may choose $S$ so that $\operatorname{End}(S) = \mathcal{O}$, where $\mathcal{O} \subseteq B$ is a maximal order. Then $A_L \simeq C_{0,L}^2 \simeq S_L^4$, so $C_{0,L} \sim_L S_L^2$. Since $\mathbb{Q}(\sqrt{2})$ splits $B$ and all maximal orders in $\mathcal{O}$ are conjugate, there exists $\alpha \in \mathcal{O}$ such that $\alpha^2 = 2$; then $\begin{pmatrix} 0 & -1 \\ 1 & \alpha \end{pmatrix} \in \mathrm{M}_2(\mathcal{O})$ is an element of order 8, so we may consider twists of $S^2$ by powers of $\psi$. Applying the theorem of Kida now to $S^2$, with (6.15)–(6.16) we conclude there are isogenies over $K$

$$\prod_{i=0}^{7} (S^2)^{(\psi^i)} \sim \operatorname{Res}_{L|K} S_L^2 \sim \operatorname{Res}_{L|K} C_{0,L} \sim \prod_{i=0}^{7} C_0^{(\psi^i)} \sim C_0^4 \times C^4. \tag{6.17}$$

Since $C_0$ is simple, we conclude that $S$ is an isogeny factor of $C$ over $K$, and $C \sim S \times S'$ for another abelian surface $S'$. We have $(S^2)^{(\psi^4)} = (S^{(\psi^4)})^2$ since $S^{(\psi^4)}$ is the quadratic twist by $\psi^4$ (corresponding to the quadratic extension $K(\sqrt{5}) \supseteq K$), so $S^{(\psi^4)}$ is also an isogeny factor of $C$ over $K$. Since $S$ is a QM abelian surface, its $L$-series is a square, so by (6.14) we have $S \not\sim S'$, and so $C \sim S \times S^{(\psi^4)}$, and thus up to labelling $L(S, s) = L(F, s)^2$ and the result follows. $\qquad\square$

*Remark* 6.18. The argument in Theorem 6.9 can evidently be generalized to more general situations.

## 7. AN EXPLICIT MODEL FOR THE BUILDING BLOCK

In light of Theorem 6.9, we now seek explicit models for the abelian surfaces $S_F$ and $S_G$ whose $L$-functions match $F$ and $G$. Because $G$ is a quadratic twist of $F$, it suffices to exhibit a model for $S_F$. We refer to Buzzard [Buz97] and Voight [Voi19, Chapter 43] as general references on abelian surfaces with quaternionic multiplication.

Let $B := \left(\dfrac{3, 5}{\mathbb{Q}}\right)$ be the quaternion algebra over $\mathbb{Q}$ of discriminant 15, and let $\mathcal{O} \subseteq B$ be a maximal order. Then there exists $\mu \in \mathcal{O}$ such that $\mu^2 + 15 = 0$, a *principal polarization* on $\mathcal{O}$. The moduli space for principally polarized abelian surfaces with QM structure by $(\mathcal{O}, \mu)$ is a Shimura curve $X_0(15, 1)$ defined over $\mathbb{Q}$ by work of Shimura [Shi67] and Deligne [Del71].

An explicit model for $X_0(15,1)$ was computed by Jordan [Jor81, Proposition 3.2.1] (see also Jordan–Livné [JL85, Table 1]) and Elkies [Elk98, (76)]): it is described by the equation

$$X_0(15,1)\colon (x^2+3)(x^2+243) + 3y^2 = 0. \tag{7.1}$$

In particular, $X_0(15,1)$ is a genus 1 curve with $X_0(15,1)(\mathbb{Q}) = X_0(15,1)(\mathbb{R}) = \emptyset$. The Jacobian of $X_0(15,1)$ is the elliptic curve over $\mathbb{Q}$ with LMFDB label 15.a5.

The quotient by the Atkin–Lehner involution $w_{15}$ gives the following map.

$$\begin{aligned} X_0(15,1) &\to X_0(15,1)/w_{15} \simeq \mathbb{P}^1 \\ j(x,y) &= x^2/81 \end{aligned} \tag{7.2}$$

The Igusa–Clebsch invariants of weights $2, 4, 6, 10$ for the 'universal' abelian surface over $X_0(15,1)/w_{15}$ are given by $(I_2 : I_4 : I_6 : I_{10})$ where

$$\begin{aligned} I_2 &= 12(j^4 + 15j^3 + 105j^2 + 125j + 10), \\ I_4 &= 45(4j+1)(j+3)^2(j-1)^4, \\ I_6 &= 9(84j^5 + 1414j^4 + 8865j^3 + 11157j^2 + 3895j + 185)(j+3)^2(j-1)^4, \\ I_{10} &= 2(j+3)^6(j-1)^{12}. \end{aligned} \tag{7.3}$$

The invariants (7.3) were computed by Elkies (2007) using K3 surfaces and by Guo–Yang [GY17, Table 4] using Borcherds products.

With this data in hand, we look for points on $X_0(15,1)$ over $K$. The base change of its Jacobian to $K$ has rank 1, so we will find infinitely many points. Searching in a small box, we find the points $(x,y) = (\pm 9/\sqrt{-7}, \pm 180/7)$ for which $j(x,y) = -1/7$ and normalized invariants $(648 : 23625 : 17474625 : -392000000)$. From these invariants, we construct a genus 2 curve $X'$ over $K = \mathbb{Q}(\sqrt{-7})$ well-defined up to twist—we confirm that the Euler factors match *up to sign* in the sense that $L_{\mathfrak{p}}(X',T) = L_{\mathfrak{p}}(f,\pm T)^2$ for good primes $\mathfrak{p}$, and this gives strong evidence that we have found a match.

Next, we identify the quadratic field $K_\delta \supseteq K$ which is associated to the twist. For a good prime $\mathfrak{p}$, write $L_{\mathfrak{p}}(X',T) = (1 - b_{\mathfrak{p}}T + \mathrm{Nm}(\mathfrak{p})T^2)$ so that $b_{\mathfrak{p}} = \mu(\mathfrak{p})a_{\mathfrak{p}}(F)$ and $\mu(\mathfrak{p}) = \pm 1$. Then $\mu(\mathfrak{p}) = -1$ if and only if $\mathfrak{p}$ is inert in $K_\delta$; searching for fields with this property supported at primes dividing the discriminant of $X'$ gives a twist $X$ for which the Euler factors match for all $\mathfrak{p}$ with $\mathrm{Nm}(\mathfrak{p}) \le 200$.

In this way, we obtain a candidate model $X$ for a curve whose Jacobian putatively corresponds to $F$; however, this model has *enormous* coefficients. So we seek to reduce these coefficients. We first apply the reduction algorithms of Bouyer–Streng [BS15] to reduce the discriminant. We would like to apply the Cremona–Stoll algorithm [SC03] to reduce the coefficients of $X$ further, but the current implementation requires a real place in the base field. Nevertheless, we can apply some reductions by hand, and then iterate. In this way,

we find the following model for $X = X_F$: $y^2 = f_F(x)$ where

$$
\begin{aligned}
f_F(x) := ( & -127747945110654443103163734556932224458 55\sqrt{-7} \\
& - 76950699888165822373247468369131393 96789)x^6 \\
+ ( & 369033561110881127499984507760692059 63943\sqrt{-7} \\
& + 7476636040743270474172885336522730375 5813)x^5 \\
+ ( & -321917153494381391057511777261488963 97205\sqrt{-7} \\
& - 160594143536856131940416520163791514 260770)x^4 \\
+ ( & 313875730633426945520672610129421224 7520\sqrt{-7} \\
& + 143070480545726568206381517716426697 366880)x^3 \\
+ ( & 872371717006802010044187864253768259 5835\sqrt{-7} \\
& - 592131957661731305545036720052223048 66040)x^2 \\
+ ( & -408065214955555266138718277062047761 9703\sqrt{-7} \\
& + 104577735708741195126251307618138452 33833)x \\
+ & 527840106295482417952525742290110846 145\sqrt{-7} \\
& - 4937131580514455349359485517406586767 43.
\end{aligned}
$$

$(7.4)$

By construction, we know that $J_F := \mathrm{Jac}(X_F)$ has $\mathrm{End}(J_F) \supseteq \mathcal{O}$. Looking at Euler factors we see that $J_F$ does not have CM, so $\mathrm{End}(J_F) = \mathcal{O}$, since $\mathcal{O}$ is a maximal order.

**Proposition 7.5.** *For all good primes $\mathfrak{p}$ we have $L_{\mathfrak{p}}(X_F, T) = L_{\mathfrak{p}}(J_F, T) = L_{\mathfrak{p}}(F, T)^2$,*

*Proof.* The proof of this proposition is a standard application of the method of Faltings–Serre, as explained by Dieulefait–Guerberoff–Pacetti [DGP10]; we only sketch some pertinent details here. Attached to $F = f_K \otimes \psi$ is a Galois representation $\rho_{F,2} \colon \mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{Q}_2^{\mathrm{al}})$. The characteristic polynomial of Frobenius at primes above 2 and 11 are given by $1 \pm T + 2T^2$ and $1 - 4T + 11T^2$, whose roots are different and do not add zero, hence the coefficient field of $\rho_{F,2}$ may be taken as $\mathbb{Q}_2$ by Taylor [Tay94, Corollary 1]. Since $J_F$ has QM (defined over $K$) by $B$ of discriminant 15 and $2 \nmid 15$, the 2-adic Tate representation $V_2 J_F$ is free of rank 1 over $B \otimes \mathbb{Q}_2 \simeq \mathrm{M}_2(\mathbb{Q}_2)$ and so affords a Galois representation $\rho_{J_F,2} \colon \mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{Q}_2)$.

The 2-division field $K(J_F[2])$ is defined by the polynomial

$$y^{12} - 6y^{11} + 21y^{10} - 50y^9 + 69y^8 - 42y^7 - 69y^6 + 210y^5 - 190y^4 + 48y^3 + 168y^2 - 160y + 64$$

(as an absolute extension); we confirm this extension is an $S_3$-extension of $K$, so the mod 2 representation $\overline{\rho}_{J_F,2} \colon \mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$ is surjective.

Looking at Euler factors, the mod 2 representation $\overline{\rho}_{F,2} \colon \mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$ attached to $F$ is also surjective. Since the conjugate of $F$ under $\mathrm{Gal}(K \mid \mathbb{Q})$ is a twist of $F$, the extension $K(J_F[2])$ is Galois over $\mathbb{Q}$ with Galois group $S_3 \rtimes C_2 \simeq D_6$, and so arises as the Galois closure of a sextic extension of $\mathbb{Q}$ unramified away from $2, 5, 7$. From the Jones–Roberts database [JR14], there are exactly 5 such extensions whose Galois closure contains

$\mathbb{Q}(\sqrt{-7})$:

$$x^6 - 3x^5 + 2x^4 - 6x^3 + 25x^2 - 19x + 8,$$

$$x^6 - x^5 - x^4 - x^3 + 2x + 2,$$

$$x^6 - x^5 + 9x^4 - 11x^3 + 16x^2 + 14x + 4,$$

$$x^6 - 2x^5 + 11x^4 - 8x^3 + 30x^2 + 24x + 8,$$

$$x^6 - x^5 + x^4 - 29x^3 + 8x^2 - 64x + 512.$$

Looking at Euler factors, we rule out all but the first one, and then confirm that its Galois closure is isomorphic to the one for $J_F$; this shows that the two mod 2 representations $\overline{\rho}_{F,2}$ and $\overline{\rho}_{J_F,2}$ are equivalent.

Since both residual representations are isomorphic with absolutely irreducible image, following Dieulefait–Guerberoff–Pacetti [DGP10, Section 2.1] we then show that fully the 2-adic representations $\rho_{F,2}$ and $\rho_{J_F,2}$ are equivalent by showing there is no obstruction to lifting. Computing a set of obstructing primes (whose Frobenius classes generate the Galois group of the maximal exponent 2 extension of $K(J_F[2])$ unramified away from $2, 5, 7$), we conclude it is enough to check that $\operatorname{tr} \rho_{F,2}(\operatorname{Frob}_{\mathfrak{p}}) = \operatorname{tr} \rho_{J_F,2}(\operatorname{Frob}_{\mathfrak{p}})$ for a primes $\mathfrak{p}$ of $K$ dividing $3, 11, 13, 17, 23, 29$. Having already checked this for all good primes $\mathfrak{p}$ up to norm $\operatorname{Nm}(\mathfrak{p}) \leq 200$, we conclude the proof. $\qquad\square$

## 8. Other rational Bianchi newforms at level (175)

As well as the newforms $F$ and $G$ with labels 2.0.7.1-30625.1-c and 2.0.7.1-30625.1-e, there are six other rational newforms of the same level, which display a variety of phenomena. All the quadratic twists mentioned here are by $\sqrt{5}$ unless otherwise specified.

- Newform 30625.1-a is the base-change of classical modular forms in $S_2(1225)$ with LMFDB labels 1225.2.a.b and 1225.2.a.d. These are $-7$-twists of each other, and associated to the elliptic curves in the isogeny classes 1225.b and 1225.d. The base-change to $K$ of the curves in both these isogeny classes all lie in the isogeny class 2.0.7.1.30625.1-a.
- Newform 30625.1-b is a base-change, but of a newform with coefficients in $\mathbb{Q}(\sqrt{2})$. The associated modular abelian variety is a surface which splits over $K$ into the product of elliptic curves, with associated elliptic curves in the isogeny class 2.0.7.1.30625.1-b, which are $\mathbb{Q}$-curves but not base-change.
- Newform 30625.1-d is also base-change, but of a newform with coefficients in $\mathbb{Q}(\sqrt{5})$ whose associated modular abelian variety is a surface which does *not* split over $K$. Hence there is no elliptic curve associated to the newform, while there is an abelian surface with QM. The newform has CM by $-35$ (so is its own quadratic twist) and is an example of the situation described in [Cre92, p. 411].
- Newform 30625.1-f is the quadratic twist of 30625.1-a so has associated elliptic curves 2.0.7.1.30625.1-f which are base-change.
- Newform 30625.1-g is the quadratic twist of 30625.1-b so has associated elliptic curves 2.0.7.1.30625.1-g which are $\mathbb{Q}$-curves but not base-change.
- Newform 30625.1-h is the quadratic twist of 1225.1-a at level $\mathfrak{N}' = (35)$. It has associated elliptic curves in isogeny class 2.0.7.1.30625.1-h which are base-changes of

those in isogeny classes 1225.h and 1225.j, and is itself the base-change of classical newforms 1225.2.a.h and 1225.2.a.j.

## 9. Relation with the Paramodular Conjecture

We conclude with an application to the Paramodular Conjecture. Recall the following conjecture due to Brumer–Kramer [BK14] and its corrigendum [BK19], as amended by Calegari.

**Conjecture 9.1.** *There is a bijection between the set of isogeny classes of QM abelian fourfolds $B$ over $\mathbb{Q}$ of conductor $N^2$ and the set of cuspidal, nonlift Siegel paramodular newforms $f$ of genus $2$, weight $2$, and level $N$ with rational Hecke eigenvalues, up to nonzero scaling. Moreover, if $B \leftrightarrow f$ in this bijection, we have the equality*

$$L(B, s) = L(f, s, \mathrm{spin})^2. \tag{9.2}$$

As with Bianchi modular forms, the frequently arising case is where $\mathrm{End}(B)_{\mathbb{Q}} \simeq \mathrm{M}_2(\mathbb{Q})$, in which case $B \sim A^2$ for $A$ an abelian surface over $\mathbb{Q}$ with $\mathrm{End}(A) = \mathbb{Z}$ with $L(A, s) = L(f, s, \mathrm{spin})$. (When $\mathrm{End}(B)_{\mathbb{Q}}$ is a division algebra, the nonexistent geometric object $A'$ such that $L(A', s) = L(f, s, \mathrm{spin})$ could be thought of as a *fake abelian surface*.)

To our Bianchi newform $F$, by theta lift one can attach a Siegel paramodular form $\Pi$, with rational eigenvalues [BDPŞ15, Theorem 4.1] such that $L(F, s) = L(\Pi, s, \mathrm{spin})$. The form $\Pi$ satisfies the hypothesis of the Paramodular Conjecture (Conjecture 9.1), so there should be a QM abelian fourfold attached to $\Pi$. Indeed, by Theorem 6.9 the Weil restriction of scalars $\mathrm{Res}_{K|\mathbb{Q}} S_F$ of $S_F$ from $K$ to $\mathbb{Q}$ gives the desired geometric object, and

$$L(\mathrm{Res}_{K|\mathbb{Q}} S_F, s) = L(S_F, K, s) = L(F, s)^2 = L(\Pi, s, \mathrm{spin})^2. \tag{9.3}$$

*Remark* 9.4. In work of Boxer–Calegari–Gee–Pilloni [BCGP18, Lemma 10.3.2], assuming standard conjectures a general argument is given to explain the existence of a geometric object attached to an automorphic form on $\mathrm{GSp}_4$ with rational Hecke eigenvalues. In concrete examples, the veracity of the standard conjectures is very hard to check; however, when both the automorphic form and (fake) abelian surface can be explicitly given, the method of Faltings–Serre can often be applied successfully in practice [BPPTVY] to establish an equality of $L$-functions, such as (9.3).

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