EXPLICIT MODULARITY OF K3 SURFACES WITH COMPLEX MULTIPLICATION OF LARGE DEGREE

EDGAR COSTA, ANDREAS-STEPHAN ELSENHANS, JÖRG JAHNEL, AND JOHN VOIGHT

Abstract. We consider the transcendental motive of three K3 surfaces X conjectured to have complex multiplication (CM). Under this assumption, we match these to explicit algebraic Hecke quasi-characters ψ_X, and CM abelian threefolds A. This provides substantial evidence that a power of A corresponds to X under the Kuga–Satake correspondence.

1. Introduction

K3 surfaces provide a rich class of objects to study in number theory and the Langlands program, testing conjectures that connect arithmetic geometry and automorphic forms through Galois representations and L-functions.

The case where the Picard number ρ achieves its maximum $\rho = 20$ (so-called *singular*) has been well-studied. Potential modularity was established through their association with algebraic Hecke quasi-characters by Shioda–Inose [SI77, §6, Theorem 6] (see also Livné [Liv95]): the transcendental cohomology has complex multiplication (CM) by an imaginary quadratic field. The general theory of K3 surfaces with complex multiplication and their fields of definition is worked out in [Riz05] and [Val23]. Over $\mathbb{Q}$, an explicit correspondence was worked out by Elkies–Schütt [ES13], rephrased in terms of classical modular forms of weight 3.

Recent efforts towards incorporating K3 surfaces into the *L-functions and Modular Forms Database (LMFDB)* [LMFDB] has renewed questions of explicit modularity for K3 surfaces, but less is known about modularity for K3 surfaces of lower Picard number. In general, a complex K3 surface X with $\rho(X) \leq 16$ does not admit a Shioda–Inose structure. Livné–Schütt–Yui [LSY10] established modularity for the finitely many K3 Delsarte surfaces (up to twist): they are quotients of Fermat surfaces (hence CM) and the matching algebraic Hecke quasi-characters arise from Jacobi sums.

This paper advances these efforts in a new direction, through computation—we hope to illustrate a mix of algorithmic methods that can be employed more generally. We remain focused on explicit examples of K3 surfaces with apparent CM of large degree (but not generated by automorphisms). Indeed, there has been recently renewed interest [BGS24] in moduli of K3 surfaces with extra Hodge endomorphisms.

Our main result matches the transcendental cohomology of certain K3 surfaces with algebraic Hecke quasi-characters, as follows. For a nice surface X over $\mathbb{Q}$, we let $\mathrm{T}(X)_{\mathbb{Q}} \subseteq \mathrm{H}^2(X, \mathbb{Q})$ be the transcendental subspace (see Section 2). For ℓ prime, we have via the comparison theorem $\mathrm{T}(X) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ and we let $\rho_{\mathrm{T}(X),\ell} \colon \mathrm{Gal}_{\mathbb{Q}} \circlearrowright (\mathrm{T}(X) \otimes \mathbb{Q}_\ell)$ be the associated Galois representation.

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Let $X = X_i$ for $i = 1, 2, 3$ be one of the following three K3 surfaces:

$$X_1\colon w^2 = xyz(x^3 - 3xy^2 + y^3 - 3x^2z - 3xyz + 9y^2z + 6yz^2 + z^3)$$
$$X_2\colon w^2 = xyz(7x^3 - 7x^2y + y^3 + 49x^2z - 21xyz - 7y^2z + 98xz^2 + 49z^3) \tag{1.1}$$
$$X_3\colon w^2 = xyz\begin{pmatrix} 49x^3 - 304x^2y + 361xy^2 + 361y^3 + 570x^2z - 2793xyz \\ +2888y^2z + 2033xz^2 - 5415yz^2 + 2299z^3 \end{pmatrix}.$$

Then $\dim_{\mathbb{Q}_\ell} \mathrm{T}(X_i) = 6$, and there is substantial numerical evidence (via 100 digit approximations to the period lattices Elsenhans–Jahnel [EJ16, §5]) that in each case, $\mathrm{T}(X_i)_{\mathbb{Q}}$ has CM by K_i, where K_i is the cyclic sextic field defined in Table 1. In fact, this CM is by the maximal order $\mathbb{Z}_{K_i}$ in each case.

Theorem 1.2. *For $i = 1, 2, 3$ and $X = X_i$, the following statements hold.*

(a) *Suppose $\mathrm{T}(X)_{\mathbb{Q}}$ has CM by K. Then for all primes ℓ,*

$$\rho_{\mathrm{T}(X),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi_X \tag{1.3}$$

where ψ_X is of ∞-type $\{(0,2),(1,1),(1,1)\}$ defined in Table 2. In particular, we have

$$L(\mathrm{T}(X), s) = L(s, \psi_X). \tag{1.4}$$

(b) *Let $A = A_i = \mathrm{Jac}(C_i)$ be the Jacobian (abelian threefold) defined in Table 1. Then*

$$\rho_{\mathrm{H}^1(A),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi_A \tag{1.5}$$

where ψ_A is of ∞-type $\{(0,1),(0,1),(0,1)\}$ defined in Table 3, and

$$L(\mathrm{H}^1(A), s) = L(s, \psi_A). \tag{1.6}$$

(c) *We have*

$$\begin{aligned} \rho_{\mathrm{H}^2(A),\ell} &\simeq \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{Q}_\ell(1) \oplus \rho_{\mathrm{T}(X),\ell} \oplus \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi' \\ L(\mathrm{H}^2(A), s) &= \zeta_F(s+1) L(\mathrm{T}(X), s) L(s, \psi'), \end{aligned} \tag{1.7}$$

where $\mathbb{Q}_\ell(1)$ is the Tate twist, $F \subseteq K$ is the unique cubic subfield, and ψ' is of ∞-type $\{(0,2),(0,2),(1,1)\}$ defined in Table 4.

This provides substantial evidence that a power of A corresponds to X under the Kuga–Satake correspondence.

Table 1 comes from Weng [Wen01, §6] and is certified correct [CMSV19]. For the fourth row ($i = 4$), we were not able to find a matching K3 surface (among double covers of $\mathbb{P}^2$ branched along 6 lines, possibly due to the nontrivial class group), but part (b) still holds; it would be interesting to produce a K3 surface in this case (not necessarily a double plane).

There is a natural Galois action on algebraic Hecke quasi-characters by $\psi^\sigma = \psi \circ \sigma$ for $\sigma \in \mathrm{Gal}(K \,|\, \mathbb{Q})$, with $L(s, \psi^\sigma) = L(s, \psi)$. Up to this natural Galois action, the characters ψ_X, ψ_A, and ψ' in Theorem 1.2 are unique.

Our computations are performed in Magma [BCP97]; the code is available online at `https://github.com/edgarcosta/K3withCM/`.

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i	LMFDB label of K_i	Defining equation for C_i
1	6.0.419904.1	$y^2 = x^7 + 6x^5 + 9x^3 + x$
2	6.0.153664.1	$y^2 = x^7 + 7x^5 + 14x^3 + 7x$
3	6.0.8340544.1	$y^2 = x^7 + 1786x^5 + 44441x^3 + 278179x$
4	6.0.59105344.1	$y^2 = x^7 + 961x^5 - 3694084x^3 + 1832265664x$

TABLE 1. Polynomials defining the CM numberfield and the genus 3 curve.

i	$\mathrm{cond}(\psi_{X_i})$	$L_p(\psi_{X_i}, T)$
1	64.1	$1 - 6 \cdot T + 15 \cdot 17 + 12 \cdot 17^2T^3 + 15 \cdot 17^3T^4 - 6 \cdot 17^4T^5 + 17^6T^6$
2	3136.1	$1 - 2 \cdot T + 19 \cdot 13 + 4 \cdot 13^2T^3 + 19 \cdot 13^3T^4 - 2 \cdot 13^4T^5 + 13^6T^6$
3	23104.1	$1 + 14 \cdot T - 5 \cdot 37 - 28 \cdot 37^2T^3 - 5 \cdot 37^3T^4 + 14 \cdot 37^4T^5 + 37^6T^6$
4	61504.13	$1 - 38 \cdot T - 9 \cdot 29 + 52 \cdot 29^2T^3 - 9 \cdot 29^3T^4 - 38 \cdot 29^4T^5 + 29^6T^6$

TABLE 2. Uniquely defining properties of ψ_X, up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

i	$\mathrm{cond}(\psi_{A_i})$	$L_p(\psi_{A_i}, T)$
1	4096.1	$1 - 6 \cdot T + 15 \cdot T^2 - 52 \cdot T^3 + 15 \cdot 17 - 6 \cdot 17^2T^5 + 17^3T^6$
2	25088.1	$1 + 4 \cdot T + 7 \cdot T^2 + 40 \cdot T^3 + 7 \cdot 13 + 4 \cdot 13^2T^5 + 13^3T^6$
3	184832.1	$1 + 4 \cdot T + 15 \cdot T^2 - 152 \cdot T^3 + 15 \cdot 37 + 4 \cdot 37^2T^5 + 37^3T^6$
4	3936256.41	$1 + 4 \cdot T + 51 \cdot T^2 + 216 \cdot T^3 + 51 \cdot 29 + 4 \cdot 29^2T^5 + 29^3T^6$

TABLE 3. Uniquely defining properties of ψ_A, up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

i	$\mathrm{cond}(\psi'_i)$	$L_p(\psi'_i, T)$
1	1.1	$1 + 42 \cdot T + 1023 \cdot T^2 + 1132 \cdot 17 + 1023 \cdot 17^2T^4 + 42 \cdot 17^4T^5 + 17^6T^6$
2	1.1	$1 + 34 \cdot T + 631 \cdot T^2 + 652 \cdot 13 + 631 \cdot 13^2T^4 + 34 \cdot 13^4T^5 + 13^6T^6$
3	1.1	$1 + 82 \cdot T + 4423 \cdot T^2 + 5452 \cdot 37 + 4423 \cdot 37^2T^4 + 82 \cdot 37^4T^5 + 37^6T^6$
4	1.1	$1 + 74 \cdot T + 3067 \cdot T^2 + 3268 \cdot 29 + 3067 \cdot 29^2T^4 + 74 \cdot 29^4T^5 + 29^6T^6$

TABLE 4. Uniquely defining properties of ψ', up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

2. SETUP

Let X be a polarized K3 surface over a number field F. We denote by X^{al} its base change to F^{al} an algebraic closure of F. We are interested in studying the Galois representations that arise from $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{al}}, \mathbb{Q}_\ell)$, for a prime ℓ. We denote the Néron–Severi group of X by $\mathrm{NS}(X)$. Under the isomorphism $\mathrm{NS}(X) \simeq \mathrm{Pic}(X)$, we may identify $\mathrm{NS}(X^{\mathrm{al}}) = \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X, \mathbb{C}) \subsetneq \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$, where $\rho(X) := \mathrm{rk}\,\mathrm{NS}(X)$ is the Picard number. Let $\mathrm{T}(X)$ be the transcendental lattice of X, the orthogonal complement of $\mathrm{NS}(X_{\mathbb{C}})$ in $\mathrm{H}^2(X, \mathbb{Z})$. The space $\mathrm{T}(X)$ is a sub-Hodge structure of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$ with Hodge numbers $(1, 20 - \rho(X), 1)$. Let $E(X)$ be the algebra of endomorphisms of $\mathrm{T}(X)$ that respect the Hodge structure. Zarhin [Zar83, Theorems 1.5.1, 1.6] has shown that $E(X)$ is either a totally real field or a CM field.

Since $\mathrm{NS}(X^{\mathrm{al}})$ and $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{al}}, \mathbb{Q}_\ell)$ come equipped with a Galois action, we also have a Galois action on $\mathrm{T}(X) \otimes \mathbb{Q}_\ell$, and the Galois representation $\rho_{\mathrm{H}^2,\ell}\colon \mathrm{Gal}(F^{\mathrm{al}} \mid F) \to \mathrm{GO}(\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes$

$\mathbb{Q}_\ell)$ decomposes as $\rho_{\mathrm{H}^2,\ell} = \rho_{\mathrm{NS},\ell} \oplus \rho_{T,\ell}$, where

$$\rho_{\mathrm{NS},\ell}\colon \operatorname{Gal}(F^{\mathrm{al}}\,|\,F) \to \operatorname{GO}(\mathrm{NS}(X)\otimes\mathbb{Q}_\ell) \quad \text{and} \quad \rho_{T,\ell}\colon \operatorname{Gal}(F^{\mathrm{al}}\,|\,F) \to \operatorname{GO}(\mathrm{T}(X)\otimes\mathbb{Q}_\ell).$$

Here we focus our attention on the Galois representation $\rho_{T,\ell}$. We define L-functions associated to the three representations [Ser70] and

$$L(\mathrm{H}^2(X), s) = L(\mathrm{NS}(X), s)L(\mathrm{T}(X), s).$$

In the case that $\dim_{E(X)} \mathrm{T}(X) = 1$, $E(X)$ is necessarily a CM field, and by class field theory we have $L(\mathrm{T}(x), s) = L(\psi, x)$ for some algebraic Hecke quasi-character ψ over $E(X)$.

We consider K3 surfaces $X \to \mathbb{P}^2$ as (resolutions of) branched over 6 lines in general (and in particular in good) position. In this case, $\rho(X) = 16$.

3. Proof of main result

Under the assumption that the L-function matches an algebraic Hecke quasi-character, to find the correct one we need to bound its conductor. It seems difficult in general to obtain such a bound by computing the conductor of the L-function of the K3 surface (though we expect it to be bounded by the discriminant of the model, defined appropriately). We can however produce a finite list of possibilities as follows. We start with the list of bad primes of the K3 surface and the primes above them in K. To bound the exponents of these primes, recall that (by the $\mathfrak{p}$-adic logarithm) the unit groups $(\mathbb{Z}_{K,\mathfrak{p}}/\mathfrak{p}^e)^\times$ as $e \to \infty$ have a bounded number of invariant factors. So to show that the exponent is bounded, we just need to show that the order of the finite part of the Hecke quasi-character is bounded. For that purpose, we note the following simple lemma.

Lemma 3.1. *Let ψ be an algebraic Hecke quasi-character over K of modulus $\mathfrak{N}$ and let $M \subset \mathbb{C}$ be the field generated by the values of ψ. Let $\chi\colon (\mathbb{Z}_K/\mathfrak{N})^\times \to \mathbb{C}^\times$ be the Dirichlet character defined by $\chi(a) = \psi(a\mathbb{Z}_K)\psi_\infty(a)$. Then $\mathbb{Q}(\chi) \subseteq M$.*

Proof. By definition, an algebraic Hecke quasi-character takes values in a number field. From the idelic formulation, we conclude that the subfield generated by the restriction of ψ to the infinite places is contained in M, hence also $\mathbb{Q}(\chi)$. □

Proof of Theorem 1.2. We first prove (a). We compute the bad primes for X by checking if the reduction no longer leads to 15 distinct intersection points of the 6 lines. For the examples above, we obtained the sets of primes $\{2, 3\}$, $\{2, 7\}$, and $\{2, 7, 11, 19\}$. We bound the exponents of the primes using Lemma 3.1, and compute the full list of algebraic Hecke quasi-characters using Magma (see work of Watkins [Wat11] for an algorithmic description) with the required ∞-type. In the style of Sherlock Holmes, we eliminate all but one (up the action of $\operatorname{Gal}(K\,|\,\mathbb{Q})$) by finding primes p uniquely identifying $L_p(\mathrm{T}(X), T) = L_p(\psi, T)$. Part (b) is proven in the same way as part (a).

For part (c), we note that $\mathrm{H}^2(A) \simeq \bigwedge^2 \mathrm{H}^1(A)$, so applying (b) and identifying characters we find that $\mathrm{H}^2(A) \simeq V_1 \oplus V_2 \oplus V_3$ as representations of $\operatorname{Gal}_{\mathbb{Q}}$, where $V_1 \simeq \operatorname{Ind}_{\operatorname{Gal}_F}^{\operatorname{Gal}_{\mathbb{Q}}} \mathbb{Q}_\ell(1)$ and $\dim_K V_2 = \dim_K V_3 = 1$. The relationship between the two characters ψ_A and ψ_X can be further encoded by the equality

$$\psi_X(\mathfrak{p}) = \psi_A(\sigma_1(\mathfrak{p}))\psi_A(\sigma_2(\mathfrak{p})) \tag{3.2}$$

for all unramified primes $\mathfrak{p}$ of degree 1 and where $\sigma_1, \sigma_2 \in \operatorname{Gal}(K\,|\,\mathbb{Q})$ are the two elements of order 3. We then finish as in (a). □

References

[BGS24] Eva Bayer-Fluckiger, Bert van Geemen, and Matthias Schütt. "K3 surfaces with real or complex multiplication". In: *preprint* (2024). `arXiv:2401.04072` (↑ 1).

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. "The Magma algebra system. I. The user language". In: *J. Symbolic Comput.* 24.3-4 (1997). Computational algebra and number theory (London, 1993), pp. 235–265 (↑ 2).

[CMSV19] Edgar Costa, Nicolas Mascot, Jeroen Sijsling, and John Voight. "Rigorous computation of the endomorphism ring of a Jacobian". In: *Math. Comp.* 88.317 (2019), pp. 1303–1339 (↑ 2).

[ES13] Noam D. Elkies and Matthias Schütt. "Modular forms and K3 surfaces". In: *Adv. Math.* 240 (2013), pp. 106–131 (↑ 1).

[EJ16] Andreas-Stephan Elsenhans and Jörg Jahnel. "Point counting on K3 surfaces and an application concerning real and complex multiplication". In: *LMS J. Comput. Math.* 19.suppl. A (2016), pp. 12–28 (↑ 2).

[Liv95] Ron Livné. "Motivic orthogonal two-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$". In: *Israel J. Math.* 92.1-3 (1995), pp. 149–156 (↑ 1).

[LSY10] Ron Livné, Matthias Schütt, and Noriko Yui. "The modularity of K3 surfaces with non-symplectic group actions". In: *Math. Ann.* 348.2 (2010), pp. 333–355 (↑ 1).

[LMFDB] The LMFDB Collaboration. "The L-functions and modular forms database". `https://www.lmfdb.org`. [Online; accessed 30 December 2024]. 2024 (↑ 1).

[Riz05] Jordan Rizov. "Complex multiplication for K3 surfaces". In: *preprint* (2005). `arXiv:1705.03054` (↑ 1).

[Ser70] Jean-Pierre Serre. "Facteurs locaux des fonctions zêta des varietés algébriques (définitions et conjectures)". In: *Séminaire Delange-Pisot-Poitou. 11e année: 1969/70. Théorie des nombres. Fasc. 1: Exposés 1 à 15; Fasc. 2: Exposés 16 à 24*. Secrétariat Math., Paris, 1970, p. 15 (↑ 4).

[SI77] T. Shioda and H. Inose. "On singular $K3$ surfaces". In: *Complex analysis and algebraic geometry*. Iwanami Shoten Publishers, Tokyo, 1977, pp. 119–136 (↑ 1).

[Val23] Domenico Valloni. "Fields of definition of K3 surfaces with complex multiplication". In: *J. Number Theory* 242 (2023), pp. 436–470 (↑ 1).

[Wat11] Mark Watkins. "Computing with Hecke Grössencharacters". In: *Actes de la Conférence "Théorie des Nombres et Applications"*. Vol. 2011. Publ. Math. Besançon Algèbre Théorie Nr. Presses Univ. Franche-Comté, Besançon, 2011, pp. 119–135 (↑ 4).

[Wen01] Annegret Weng. "A class of hyperelliptic CM-curves of genus three". In: *J. Ramanujan Math. Soc.* 16.4 (2001), pp. 339–372 (↑ 2).

[Zar83] Yu. G. Zarhin. "Hodge groups of $K3$ surfaces". In: *J. Reine Angew. Math.* 341 (1983), pp. 193–220 (↑ 3).

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave., MA 02139, USA
Email address: edgarc@mit.edu
URL: https://edgarcosta.org

School of Mathematics and Statistics, Carslaw Building (F07), University of Sydney, NSW 2006, Australia

Institut für Mathematik, Emil-Fischer Strasse 30, D-97074 Wüerzburg, Germany
Email address: Stephan.Elsenhans@uni-wuerzburg.de
URL: https://www.mathematik.uni-wuerzburg.de/computeralgebra/team/elsenhans-stephan-prof-dr/

Department Mathematik, Univ. Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany
Email address: jahnel@mathematik.uni-siegen.de
URL: http://www.uni-math.gwdg.de/jahnel

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA

School of Mathematics and Statistics, Carslaw Building (F07), University of Sydney, NSW 2006, Australia
Email address: jvoight@gmail.com
URL: http://jvoight.github.io/