Contemporary Mathematics

Explicit modularity of K3 surfaces with complex multiplication of large degree

Edgar Costa, Andreas-Stephan Elsenhans, Jörg Jahnel, and John Voight

Abstract. We consider the transcendental motives of three K3 surfaces X conjectured to have complex multiplication (CM). Under this assumption, we match these to explicit algebraic Hecke quasi-characters ψ_X, and CM abelian threefolds A. This provides substantial evidence that a power of A corresponds to X under the Kuga–Satake correspondence.

1. Introduction

K3 surfaces provide a rich class of objects to study in number theory and the Langlands program, testing conjectures that connect arithmetic geometry and automorphic forms through Galois representations and L-functions.

The case where the Picard number ρ achieves its maximum $\rho = 20$ has been well-studied. Potential modularity was established through their association with algebraic Hecke quasi-characters (also called Hecke Grossencharacters or just Hecke characters) by Shioda–Inose [SI77, §6, Theorem 6] (see also Livné [Liv95]): the transcendental cohomology has complex multiplication (CM) by an imaginary quadratic field. Over $\mathbb{Q}$, an explicit correspondence with classical modular forms of weight 3 was worked out by Elkies–Schütt [ES13].

Recent efforts towards incorporating K3 surfaces into the *L-functions and Modular Forms Database (LMFDB)* [LMFDB] has renewed questions of explicit modularity for K3 surfaces, but less is known about modularity for K3 surfaces of lower Picard number. In general, a complex K3 surface X with $\rho(X) \leq 16$ does not admit a Shioda–Inose structure. Piatetski-Shapiro–Shafarevich [PŠ73] expressed the L-function of a K3 surfaces with complex multiplication as a product of Hecke L-functions over some finite extension via the Kuga–Satake correspondence and applying the corresponding statement for abelian varieties. The theory of complex multiplication for K3 surfaces was further developed by Rizov [Riz05]. Building on this work, Valloni [Val23] considers K3 surfaces with CM by the full ring of integers and studied their fields of definition; and more recently, Ito [Ito25] is more explicit about the properties of the Hecke quasi-characters that appear in the equality of L-series. Livné–Schütt–Yui [LSY10] established modularity for the finitely many K3 Delsarte surfaces (up to twist): they are (CM) quotients of Fermat surfaces.

This paper advances these efforts in a new direction, through computation. We remain focused on explicit examples of K3 surfaces with apparent CM of large degree. Indeed, there has been recently renewed interest [BGS24] in moduli of K3 surfaces with extra Hodge endomorphisms. Our main result matches the transcendental cohomology of certain K3 surfaces with algebraic Hecke quasi-characters, as follows. For a complex surface X, we let $\mathrm{T}(X)_{\mathbb{Q}} \subseteq \mathrm{H}^2(X,\mathbb{Q})$ be the transcendental subspace (see section 2). If moreover X is defined over a number field $F \subset \mathbb{C}$, then for ℓ prime, we have via comparison $\mathrm{T}(X) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(X,\mathbb{Q}_\ell)$ and we let $\rho_{\mathrm{T}(X),\ell}\colon \mathrm{Gal}_F \circlearrowright (\mathrm{T}(X)\otimes\mathbb{Q}_\ell)$ be the associated Galois representation.

Let $X = X_i$ for $i = 1,2,3$ be one of the three K3 surfaces obtained from the following affine models:

$$\begin{aligned} &X_1\colon w^2 = xyz(x^3 - 3xy^2 + y^3 - 3x^2z - 3xyz + 9y^2z + 6yz^2 + z^3) \\ &X_2\colon w^2 = xyz(7x^3 - 7x^2y + y^3 + 49x^2z - 21xyz - 7y^2z + 98xz^2 + 49z^3) \\ &X_3\colon w^2 = xyz\begin{pmatrix} 49x^3 - 304x^2y + 361xy^2 + 361y^3 + 570x^2z - 2793xyz \\ +2888y^2z + 2033xz^2 - 5415yz^2 + 2299z^3 \end{pmatrix}. \end{aligned} \tag{1.1}$$

More precisely, we take X_i to be the smooth projective surface obtained from the taking branched double cover of $\mathbb{P}^2$ defined by (1.1) and blowing up the $15 = \binom{6}{2}$ double points in the branch locus of 6 lines. Then $\dim_{\mathbb{Q}_\ell} \mathrm{T}(X_i) = 22 - 16 = 6$, and there is substantial numerical evidence that in each case, $\mathrm{T}(X_i)_{\mathbb{Q}}$ has CM by K_i, where $K_i = F_i(\sqrt{-1})$ is the cyclic sextic field defined in Table 1 by their LMFDB label. For this evidence, see Elsenhans–Jahnel [EJ16, §5] and the end of section 2.

Theorem 1.2. *For $i = 1,2,3$ and $X = X_i$, the following statements hold.*

(a) *Suppose $\mathrm{T}(X)_{\mathbb{Q}}$ has CM by K. Then for all primes ℓ,*

$$\rho_{\mathrm{T}(X),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi_X \tag{1.3}$$

where ψ_X is of ∞-type $\{(0,2),(1,1),(1,1)\}$ defined in Table 2. In particular, we have

$$L(\mathrm{T}(X),s) = L(s,\psi_X). \tag{1.4}$$

(b) *Let $A = A_i = \mathrm{Jac}(C_i)$ be the Jacobian defined in Table 1. Then*

$$\begin{aligned} \rho_{\mathrm{H}^1(A),\ell} &\simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi_A \\ L(\mathrm{H}^1(A),s) &= L(s,\psi_A) \end{aligned} \tag{1.5}$$

where ψ_A is of ∞-type $\{(0,1),(0,1),(0,1)\}$ defined in Table 3.

(c) *We have*

$$\begin{aligned} \rho_{\mathrm{H}^2(A),\ell} &\simeq \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{Q}_\ell(1) \oplus \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} (\psi_X \oplus \psi') \\ L(\mathrm{H}^2(A),s) &= \zeta_F(s+1)L(s,\psi_X)L(s,\psi'), \end{aligned} \tag{1.6}$$

where $\mathbb{Q}_\ell(1)$ is the Tate twist, $F \subseteq K$ is the unique cubic subfield, and ψ' is of ∞-type $\{(0,2),(0,2),(1,1)\}$ defined in Table 4.

This provides substantial evidence that a power of A corresponds to X under the Kuga–Satake correspondence [KS67]: for more, see Remark 3.2. Our computations are performed in Magma [BCP97]; the code is available at `https://github.com/edgarcosta/K3withCM/`. There is a natural Galois action on algebraic Hecke quasi-characters by $\psi^\sigma = \psi \circ \sigma$ for $\sigma \in \mathrm{Gal}(K\,|\,\mathbb{Q})$, with $L(s,\psi^\sigma) = L(s,\psi)$. Up to this Galois action, the characters ψ_X, ψ_A, and ψ' in Theorem 1.2 are unique.

i	K_i	F_i	Defining equation for C_i
1	6.0.419904.1	3.3.81.1	$y^2 = x^7 + 6x^5 + 9x^3 + x$
2	6.0.153664.1	3.3.49.1	$y^2 = x^7 + 7x^5 + 14x^3 + 7x$
3	6.0.8340544.1	3.3.361.1	$y^2 = x^7 + 1786x^5 + 44441x^3 + 278179x$
4	6.0.59105344.1	3.3.961.1	$y^2 = x^7 + 961x^5 - 3694084x^3 + 1832265664x$

TABLE 1. Polynomials defining CM fields and genus 3 curves.

i	$\mathrm{cond}(\psi_{X_i})$	p	$L_p(\psi_{X_i}, T)$
1	64.1	17	$1 - 6T + 15pT^2 + 12p^2T^3 + 15p^3T^4 - 6p^4T^5 + p^6T^6$
2	3136.1	13	$1 - 2T + 19pT^2 + 4p^2T^3 + 19p^3T^4 - 2p^4T^5 + p^6T^6$
3	23104.1	37	$1 + 14T - 5pT^2 - 28p^2T^3 - 5p^3T^4 + 14p^4T^5 + p^6T^6$
4	61504.13	29	$1 - 38T - 9pT^2 + 52p^2T^3 - 9p^3T^4 - 38p^4T^5 + p^6T^6$

TABLE 2. Uniquely defining properties of ψ_X, up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

i	$\mathrm{cond}(\psi_{A_i})$	p	$L_p(\psi_{A_i}, T)$
1	4096.1	17	$1 - 6T + 15T^2 - 52T^3 + 15pT^4 - 6p^2T^5 + p^3T^6$
2	25088.1	13	$1 + 4T + 7T^2 + 40T^3 + 7pT^4 + 4p^2T^5 + p^3T^6$
3	184832.1	37	$1 + 4T + 15T^2 - 152T^3 + 15pT^4 + 4p^2T^5 + p^3T^6$
4	3936256.41	29	$1 + 4T + 51T^2 + 216T^3 + 51pT^4 + 4p^2T^5 + p^3T^6$

TABLE 3. Uniquely defining properties of ψ_A, up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

Remark 1.7. To each Hecke quasi-character ψ for a CM extension $K \supseteq F$, by restriction of the automorphic representation we can also associate a Hilbert modular form f over F with matching Galois representation and L-function. As such forms f have nontrivial character (and in parts (a) and (c), weights $(3, 3, 1)$ and $(3, 1, 1)$, respectively), they currently fall outside the database of Hilbert modular forms in the LMFDB. We hope to see them in a future expansion of the database.

Table 1 comes from Weng [Wen01, §6] and is certified correct [CMSV19]. For the fourth row ($i = 4$), we were not able to find a matching K3 surface (among double covers of $\mathbb{P}^2$ branched along 6 lines, possibly due to the nontrivial class group), but part (b) still holds; it would be interesting to produce a K3 surface in this case (not necessarily a degree 2 model).

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2. Setup

Let X be a polarized K3 surface over a number field F. We denote by X^{al} its base change to F^{al}, an algebraic closure of F. We are interested in studying the Galois representations that arise from $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{al}}, \mathbb{Q}_\ell)$, for a prime ℓ. Let $\mathrm{NS}(X)$ denote the Néron–Severi group of X. Under the canonical isomorphism $\mathrm{NS}(X) \cong \mathrm{Pic}(X)$, we may identify $\mathrm{NS}(X^{\mathrm{al}}) \cong \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X, \mathbb{C}) \subsetneq \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$. Let $\rho(X) := \mathrm{rk}\, \mathrm{NS}(X)$ be the Picard number.

i	$\mathrm{cond}(\psi_i')$	p	$L_p(\psi_i', T)$
1	1.1	17	$1 + 42T + 1023T^2 + 1132pT^3 + 1023p^2T^4 + 42p^4T^5 + p^6T^6$
2	1.1	13	$1 + 34T + 631T^2 + 652pT^3 + 631p^2T^4 + 34p^4T^5 + p^6T^6$
3	1.1	37	$1 + 82T + 4423T^2 + 5452pT^3 + 4423p^2T^4 + 82p^4T^5 + p^6T^6$
4	1.1	29	$1 + 74T + 3067T^2 + 3268pT^3 + 3067p^2T^4 + 74p^4T^5 + p^6T^6$

TABLE 4. Uniquely defining properties of ψ', up to $\mathrm{Gal}(K \mid \mathbb{Q})$.

Let $\mathrm{T}(X)$ be the transcendental lattice of X, the orthogonal complement of $\mathrm{NS}(X_{\mathbb{C}})$ in $\mathrm{H}^2(X, \mathbb{Z})$. The space $\mathrm{T}(X)$ is a sub-Hodge structure of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})$ with Hodge numbers $(1, 20 - \rho(X), 1)$. Let $E = E(X)$ be the algebra of endomorphisms of $\mathrm{T}(X)$ that respect the Hodge structure. Zarhin [Zar83, Theorems 1.5.1, 1.6] has shown that E is either a totally real field or a CM field.

The Galois representation $\rho_{\mathrm{H}^2,\ell} \colon \mathrm{Gal}(F^{\mathrm{al}} \mid F) \to \mathrm{GO}(\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Q}_\ell)$ decomposes as $\rho_{\mathrm{H}^2,\ell} = \rho_{\mathrm{NS},\ell} \oplus \rho_{\mathrm{T},\ell}$, and we focus on

$$\rho_{\mathrm{T},\ell} \colon \mathrm{Gal}(F^{\mathrm{al}} \mid F) \to \mathrm{GO}(\mathrm{T}(X) \otimes \mathbb{Q}_\ell).$$

and its associated [Ser70] L-function $L(\mathrm{T}(X), s)$. In the case that $\dim_E \mathrm{T}(X) = 1$, in fact E is necessarily a CM field, and by class field theory we have $L(\mathrm{T}(X), s) = L(s, \psi)$ for some algebraic Hecke quasi-character ψ over E.

We consider K3 surfaces $X \to \mathbb{P}^2$ as (resolutions of) branched over 6 lines in general (and in particular in good) position. In this case, $\rho(X) \geq 16$, with equality when the lines are in very general position. We restrict to the K3 surfaces identified in (1.1). As mentioned in the introduction, there is strong numerical evidence [EJ16, §5] that these K3 surfaces have complex multiplication (CM). We further computed 100 digit approximations to the period lattices using the method of Elsenhans–Jahnel [EJ24, §6]. In fact, this CM is apparently by the maximal order $\mathbb{Z}_{K_i}$ in each case. More precisely, for each surface we found numerical approximations of six period integrals $\tau_1, \ldots, \tau_6$ that form a basis of the period lattice such that the ratios τ_i/τ_1 for $i = 1, \ldots, 6$ coincide with a $\mathbb{Z}$-basis of the maximal order of the conjectural endomorphism field. For the surface X_1 and for chosen cycles,

$$(\tau_1, \ldots, \tau_6) \approx (2.6402, 11.6474, 7.60232, -7.6023i, -4.96206i, -6.68537i);$$

with respect to the eigenvalues 0.467911 and i, the period vector is an eigenvector of $\begin{pmatrix} 2 & -1 & 1 & & & \\ -1 & 2 & -2 & & & \\ 0 & -1 & 2 & & & \\ & & & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -2 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} & & & -1 & 1 & 0 \\ & & & 0 & -1 & -1 \\ & & & -1 & 0 & 0 \\ 0 & 0 & 1 & & & \\ -1 & 0 & 1 & & & \\ 1 & 1 & -1 & & & \end{pmatrix}$ and the cup form is $\begin{pmatrix} -1 & 0 & 1 & & & \\ 0 & 0 & 1 & & & \\ 1 & 1 & -1 & & & \\ & & & 0 & 1 & 1 \\ & & & 1 & -1 & 0 \\ & & & 1 & 0 & 0 \end{pmatrix}$. The data for the other examples are very similar. Computing periods to a precision of 100 decimal places took about half an hour on a standard desktop.

3. Proof of main result

Under the assumption that the L-function matches an algebraic Hecke quasi-character, to find the correct one we need to bound its conductor. It seems difficult in general to obtain such a bound by computing the conductor of the L-function of the K3 surface. We can however produce a finite list of possibilities as follows. We start with the list of bad primes of the K3 surface and the primes above them in K. To bound the exponents of these primes, recall that (by the $\mathfrak{p}$-adic logarithm) the unit groups $(\mathbb{Z}_{K,\mathfrak{p}}/\mathfrak{p}^e)^\times$ as $e \to \infty$ have a bounded number of invariant factors. So to show that the exponent is bounded, we just need to show that the order of the finite part of the Hecke quasi-character is bounded, using the following lemma.

Lemma 3.1. *Let ψ be an algebraic Hecke quasi-character over K of modulus $\mathfrak{N}$ and let $M \subset \mathbb{C}$ be the field generated by the values of ψ. Let $\chi\colon (\mathbb{Z}_K/\mathfrak{N})^\times \to \mathbb{C}^\times$ be the Dirichlet character defined by $\chi(a) = \psi(a\mathbb{Z}_K)\psi_\infty(a)$. Then $\mathbb{Q}(\chi) \subseteq M$.*

Proof. By definition, an algebraic Hecke quasi-character takes values in a number field. From the idelic formulation, we conclude that the subfield generated by the restriction of ψ to the infinite places is contained in M, hence also $\mathbb{Q}(\chi)$. □

Proof of Theorem 1.2. We first prove (a). We compute the bad primes for X by checking if the reduction no longer leads to 15 distinct intersection points of the 6 lines. We bound the exponents of the primes using Lemma 3.1. Following Watkins [Wat11, § 5.2], using Magma we compute the full list of algebraic Hecke quasi-characters ψ with the required ∞-type, conductor bounded as above, and $\mathbb{Q}(\psi) \subseteq K_i$. More precisely, we start with the principal character ψ_0 of the chosen ∞-type and its associated Dirichlet character χ_0 (see Lemma 3.1). Next, we enumerate the Dirichlet characters χ whose lifts to Hecke characters twist ψ_0 to give a primitive character ψ with $\mathbb{Q}(\psi) \subseteq K_i$. Concretely, we require that $\chi' := \chi/\chi_0$ be primitive, trivial on units, and satisfy $\mathbb{Q}(\chi) \subseteq K_i$. Because all these conditions can be phrased on the abstract character group, we apply the filters there rather than iterating over every element, a task that would be impractical for large levels. For example, for X_3 we consider characters of conductor $\mathfrak{N} = \mathfrak{p}_2^7 \cdot 7 \cdot 11 \cdot \mathfrak{p}_{19}$, where $\mathrm{Nm}(\mathfrak{p}_p) = p$. The Dirichlet character group modulo $\mathfrak{N}$ is isomorphic to

$$(\mathbb{Z}/4\mathbb{Z})^5 \oplus (\mathbb{Z}/8\mathbb{Z})^2 \oplus (\mathbb{Z}/24\mathbb{Z})^2 \oplus \mathbb{Z}/48\mathbb{Z} \oplus (\mathbb{Z}/240\mathbb{Z})^3 \oplus \mathbb{Z}/5040\mathbb{Z}$$

which contains over 10^{20} elements; of these, only 279 936 satisfy our requirements.

We then compute $L_p,(\mathrm{T}(X), T)$ using a method of Elsenhans–Jahnel [EJ16] based on a trace formula involving a matrix expansion. In the style of Sherlock Holmes, we eliminate all but one (up to the action of $\mathrm{Gal}(K \mid \mathbb{Q})$) by finding primes p uniquely identifying $L_p(\mathrm{T}(X), T) = L_p(\psi, T)$. It was enough to consider good primes $p < 250$ totally split in K_i to obtain a unique match for each example in a single pass. After a match was found, we identified a prime p which, together with the conductor, uniquely identifies the character (up to the Galois action).

Part (b) is proven in the same way as part (a). For part (c), we note that $\mathrm{H}^2(A) \simeq \bigwedge^2 \mathrm{H}^1(A)$, so applying (b) and identifying characters we find that $\mathrm{H}^2(A) \simeq V_1 \oplus V_2 \oplus V_3$ as representations of $\mathrm{Gal}_{\mathbb{Q}}$, where $V_1 \simeq \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{Q}_\ell(1)$ and $\dim_K V_2 = \dim_K V_3 = 1$. The relationship between the two characters ψ_A and ψ_X can be further encoded by the equality $\psi_X(\mathfrak{p}) = \psi_A(\sigma_1(\mathfrak{p}))\psi_A(\sigma_2(\mathfrak{p}))$ for all unramified primes $\mathfrak{p}$ of degree 1 and where $\sigma_1, \sigma_2 \in \mathrm{Gal}(K \mid \mathbb{Q})$ are the two elements of order 3. We finish as in (a), checking on distinguishing primes. □

Remark 3.2. The Kuga–Satake construction [KS67] (see also van Geemen [Gee08, §5]) attaches to a complex polarized K3 surface X a complex abelian variety such that there is an embedding $\mathrm{T}(X)(1) \hookrightarrow H^1(A) \otimes H^1(A)$ of Hodge structures. In our case, this relationship is made explicit in Theorem 1.2(c) via comparison, in the sense that X and A have associated to Hecke characters ψ_X and ψ_A, with ψ_X appearing as a symmetric product of ψ_A. This strongly suggests that X and A are connected via the Kuga–Satake construction, at least up to isogeny and powers.

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Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave., MA 02139, USA

Email address: edgarc@mit.edu

URL: https://edgarcosta.org

School of Mathematics and Statistics, Carslaw Building (F07), University of Sydney, NSW 2006, Australia

Institut für Mathematik, Emil-Fischer Strasse 30, D-97074 Würzburg, Germany

Email address: Stephan.Elsenhans@uni-wuerzburg.de

URL: https://www.mathematik.uni-wuerzburg.de/computeralgebra/team/elsenhans-stephan-prof-dr/

Department Mathematik, Univ. Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany

Email address: jahnel@mathematik.uni-siegen.de

URL: http://www.uni-math.gwdg.de/jahnel

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA

School of Mathematics and Statistics, Carslaw Building (F07), University of Sydney, NSW 2006, Australia

Email address: jvoight@gmail.com

URL: https://jvoight.github.io/