UNEXPECTED PRIMES OF GOOD REDUCTION IN QUOTIENTS OF MODULAR AND SHIMURA CURVES

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Abstract. We find all zero-dimensional spaces of newforms of weight 2 and squarefree level N with a fixed Atkin–Lehner sign pattern. As an application, we classify unexpected primes of good reduction of Atkin–Lehner quotients of modular and Shimura curves of squarefree levels.

1. Introduction

Modular curves of genus zero, including their Atkin–Lehner quotients, have played a significant role in arithmetic geometry and even in group theory through monstrous moonshine. Since the genus of a modular curve is the dimension of the space of cuspforms of weight 2, in this paper we are led to the more general question: which spaces of newforms *with fixed Atkin–Lehner signs* are zero?

Let N be squarefree and let $W_N := \{w_d : d \mid N\}$ denote its Atkin–Lehner group, the elementary abelian 2-group with cardinality $\#W_N = 2^{\omega(N)}$. A *sign pattern* for N is a character

$$\varepsilon_N \colon W_N \to \{\pm 1\},$$

determined by its values on w_p for $p \mid N$. Let

$$S_2^{\mathrm{new},\varepsilon_N}(N) := \{f \in S_2^{\mathrm{new}}(\Gamma_0(N)) : w_d f = \varepsilon_N(d) f \text{ for all } d \mid N\}$$

be the Atkin–Lehner ε_N-eigenspace over $\mathbb{C}$ of classical newforms of level N and trivial character. We call the pair (N, ε_N) *lacking* if $S_2^{\mathrm{new},\varepsilon_N}(N) = \{0\}$.

A basic dimension estimate already quickly shows that there are only finitely many lacking pairs. Indeed, the main term in the trace formula for $\dim S_2^{\mathrm{new},\varepsilon_N}(N)$ grows like $N/2^{\omega(N)}$, whereas the Atkin–Lehner trace terms are controlled by class numbers of imaginary quadratic fields and have size roughly $O(\sqrt{N}\log N)$ after summing over divisors. Thus every sign pattern occurs once N is sufficiently large.

Our first result is to make this effective and then compute the complete list.

Theorem 1.1. *Let N be squarefree and let ε_N be a sign pattern for N. Then the pairs (N, ε_N) for which $S_2^{\mathrm{new},\varepsilon_N}(N) = \{0\}$ are precisely the pairs listed in Tables 2, 3, 4, and 5.*

We verified using the LMFDB [LMF20] that up to $N \leq 10000$ our tables match. The proof of Theorem 1.1 occupies Section 2; we use a version of the trace formula adapted for this purpose by Martin [Mar18]. Weinstein [Wei09] proves a result in the same spirit for Hilbert modular forms with prescribed local ramification: up to finitely many exceptions, all prescribed local behaviours occur. However, prescribing the Atkin–Lehner signs creates smaller spaces, with additional class-number terms in the trace formula; controlling these terms uniformly is the main analytic task here, and then evaluating the remaining finite range is the main computational task.

Our motivation for Theorem 1.1 comes from the following example. The curve $X_0(194)$ has bad reduction at 2 but its Atkin–Lehner quotient $X_0^*(194) = X_0(194)/W_{194}$ of genus 3 has *good* reduction at 2, and therefore so does its Jacobian! Our second result classifies all such unexpected (or gained) primes of good reduction for Atkin–Lehner quotients of Shimura curves of squarefree level.

Theorem 1.2. *Let D, N be coprime squarefree integers with $\omega(D)$ even (allowing $D = 1$), and let $X_0^*(D;N) := X_0(D;N)/W_{DN}$ be the full Atkin–Lehner quotient of the modular or Shimura curve of discriminant D and Eichler level N. Then the following statements hold.*

(a) *The curve $X_0^*(D;N)$ has good reduction at p if and only if its Jacobian has good reduction at p.*
(b) *If $X_0^*(D;N)$ has genus > 0 and $p \mid D$, then $X_0^*(D;N)$ has bad reduction at p.*
(c) *The triples (D, N, p) for primes $p \mid DN$ such that $g(X_0^*(D;N)) > 0$ and $X_0^*(D;N)$ has good reduction at p are precisely those listed in Tables 6 and 7.*

Indeed, the Jacobian $\mathrm{Jac}(X_0^*(D;N))$ decomposes up to isogeny over $\mathbb{Q}$ indexed by newforms of level dividing DN that are new at D with Atkin–Lehner eigenvalues specified by the sign pattern ε for DN with $\varepsilon(w_p) = -1, +1$ according as $p \mid D$ or $p \mid N$. Section 3 explains this criterion and then deduces Theorem 1.2 using Theorem 1.1.

The calculations in this paper deliberately focus on the cleanest case, the one relevant to our application: squarefree level, weight 2, and trivial character. Several natural extensions should be possible. The first is to allow arbitrary (nonsquarefree) level: there are only additional technicalities, with a more complicated trace formula and more local bookkeeping. The second would be to allow arbitrary nebentypus character, including signs at primes where the local component of the character χ is at most quadratic. Third, we could allow higher weight $k \geq 3$; the analysis is similar and a bit easier, with main term now the larger $(k-1)(N/2^{\omega(N)})$. Finally, one can ask for the analogous result for Hilbert modular forms over totally real fields, including a decomposition according to class group characters.

All computations implemented in this paper are performed in Magma [BCP97]. We used a database of class numbers of imaginary quadratic extensions from LMFDB [LMF20] and Kimball Martin's implementation of the trace formula [Mar18]. We refer to the GitHub repository [PPV26] for all related code, computation time, and outputs.

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2. Dimensions of weight 2 newforms

In this section, we prove Theorem 1.1. Let $N \in \mathbb{Z}_{\geq 1}$ be squarefree and let ε_N be a sign pattern for N. For $n \in \mathbb{Z}_{\geq 1}$, let $\omega(n)$ be the number of distinct prime factors of n, and let $\omega'(n)$ be the number of distinct odd prime factors of n.

Dimension formula. We use explicit formulas for dimensions of newspaces with prescribed Atkin–Lehner signs from Martin [Mar18]. For $d > 0$ squarefree, let Δ_d be the discriminant of $K_d := \mathbb{Q}(\sqrt{-d})$, and let $h'(\Delta_d)$ be its modified class number, the usual class number of the ring of integers unless $d = 1, 3$, where instead $h'(\Delta_1) = 1/2$ and $h'(\Delta_3) = 1/3$.

Proposition 2.1. *We have*

$$\begin{aligned}\dim S_2^{\mathrm{new},\varepsilon_N}(N) \geq \frac{1}{2^{\omega(N)}}\Bigg[&\dim S_2^{\mathrm{new}}(N) - 2^{\omega(N)} \\ &- \delta[2 \mid N]\left(h'(\Delta_2)|b(2, N/2)|2^{\omega(N/2)-1} + \frac{1}{2}2^{\omega(N/2)}\right) \\ &- \delta[3 \mid N]\left(h'(\Delta_3)|b(3, N/3)|2^{\omega(N/3)-1} + \frac{2}{3}2^{\omega(N/3)}\right) \\ &- \frac{1}{2}\sum_{\substack{d>3\\ d|N}} |b(d, N/d)|h'(\Delta_d)2^{\omega'(N/d)}\Bigg],\end{aligned} \tag{2.2}$$

where $\delta[P] = 1$ if P is true and 0 otherwise, and

$$b(m, n) := \begin{cases} 1, & \text{if } m \equiv 1, 2, 5, 6 \pmod 8 \text{ and } n \equiv 1 \pmod 2; \\ -1, & \text{if } m \equiv 1, 2, 5, 6 \pmod 8 \text{ and } n \equiv 0 \pmod 2; \\ 4, & \text{if } m \equiv 3 \pmod 8 \text{ and } n \equiv 1 \pmod 2; \\ -2, & \text{if } m \equiv 3 \pmod 8 \text{ and } n \equiv 0 \pmod 2; \\ 2, & \text{if } m \equiv 7 \pmod 8 \text{ and } n \equiv 1 \pmod 2; \\ 0, & \text{if } m \equiv 7 \pmod 8 \text{ and } n \equiv 0 \pmod 2. \end{cases} \tag{2.3}$$

Proof. Combine Martin [Mar18, Proposition 1.4, Proposition 3.2, Theorem 3.3], taking absolute values of the trace terms which can appear with either sign. □

The main term in Proposition 2.1 is the dimension of the full newspace.

Lemma 2.4. *For $N > 1$ we have*

$$\dim S_2^{\mathrm{new}}(N) = \frac{\varphi(N)}{12} - \frac{e_4}{4} - \frac{e_3}{3} + \mu(N),$$

where $e_D = \prod_{p|N}\left(\left(\frac{-D}{p}\right) - 1\right)$ for $D = 3, 4$, and μ is the Möbius function.

Proof. See Martin [Mar18, Theorem 2.1] (citing [Mar05, Theorem 4]). □

Estimates. We now isolate the estimates used to make the finite computation effective.

Lemma 2.5. *Let $m = \omega(N)$. Define*

$$A(N) := \begin{cases} \frac{19}{12}2^m + 1, & \text{if } N \equiv \pm 1 \pmod 6; \\ \frac{9}{4}2^m + 1, & \text{if } N \equiv 3 \pmod 6; \\ 3 \cdot 2^m + 1, & \text{if } N \equiv 0 \pmod 2. \end{cases} \tag{2.6}$$

Then for every sign pattern ε_N,

$$\dim S_2^{\mathrm{new},\varepsilon_N}(N) \geq \frac{1}{2^m}\left(\frac{\varphi(N)}{12} - A(N) - \frac{2}{\pi}\sum_{\substack{d>1\\ d\mid N}} \sqrt{d}\log(4d)2^{\omega(N/d)}\right). \tag{2.7}$$

Proof. Substitute Lemma 2.4 into Proposition 2.1. Since $\mu(N) \geq -1$ and

$$|e_D| = \prod_{p\mid N}\left|\left(\frac{-D}{p}\right) - 1\right| \leq 2^m \tag{2.8}$$

the contribution from the full newspace formula, together with the term -2^m in Proposition 2.1, is bounded below by

$$\frac{\varphi(N)}{12} - \left(1 + \frac{1}{4} + \frac{1}{3}\right)2^m - 1 = \frac{\varphi(N)}{12} - \frac{19}{12}2^m - 1. \tag{2.9}$$

This gives the first case of $A(N)$, because then neither the $d = 2$ nor the $d = 3$ exceptional term occurs.

If $N \equiv 3 \pmod 6$, then $3 \mid N$ and $2 \nmid N$. The $d = 3$ term contributes at most

$$h'(\Delta_3)|b(3, N/3)|2^{m-2} + \frac{2}{3}2^{m-1} = \frac{1}{3}\cdot 4 \cdot 2^{m-2} + \frac{2}{3}2^{m-1} = \frac{2}{3}2^m,$$

using $h'(\Delta_3) = 1/3$ and $|b(3, N/3)| = 4$. Adding this to $\frac{19}{12}2^m + 1$ gives $\frac{9}{4}2^m + 1$. Similarly, if N is even then the $d = 2$ term contributes at most 2^{m-1} since $h'(\Delta_2) = 1$ and $|b(2, N/2)| = 1$; if $3 \mid N$ then the $d = 3$ term contributes at most

$$h'(\Delta_3)|b(3, N/3)|2^{m-2} + \frac{2}{3}2^{m-1} = \frac{1}{3}\cdot 2 \cdot 2^{m-2} + \frac{2}{3}2^{m-1} = \frac{1}{2}2^m.$$

If $3 \nmid N$, there is no contribution. Thus the total coefficient of 2^m is at most

$$\frac{19}{12} + \frac{1}{2} + \frac{1}{2} = \frac{31}{12} < 3,$$

which gives the stated even case.

To control the remaining sum, from (2.3), we have $|b(d, N/d)| \leq 4$; also $\omega'(N/d) \leq \omega(N/d)$. The Louboutin bound [Lou04, Corollary 2]

$$h(\Delta_d) = h'(\Delta_d) \leq \frac{\sqrt{d}}{\pi}\log(4d) \tag{2.10}$$

for $d > 3$ therefore gives

$$\frac{1}{2}|b(d, N/d)|h'(\Delta_d)2^{\omega'(N/d)} \leq \frac{2}{\pi}\sqrt{d}\log(4d)2^{\omega(N/d)}. \tag{2.11}$$

Extending the resulting sum from $d > 3$ to $d > 1$ only increases the error term, so (2.7) follows. □

Lemma 2.12. *Let $m = \omega(N)$ and put*

$$\kappa(N) := \begin{cases} 1, & \text{if } N \equiv \pm 1 \pmod 6; \\ 2, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\substack{d>1\\ d\mid N}} \sqrt{d}\log(4d)2^{\omega(N/d)} \le \kappa(N)2^{m-1}\sqrt{N}\,(\log N + 4\log 2)\,. \tag{2.13}$$

Proof. Fix j with $0 \le j \le m-1$, and sum over divisors $d \mid N$ with $d > 1$ and $\omega(N/d) = j$. Write $e = N/d$. Then e is squarefree and has j prime factors.

We first prove

$$2^j\sqrt{d} = \sqrt{4^j d} \le \kappa(N)\sqrt{N}. \tag{2.14}$$

If $N \equiv \pm 1 \pmod 6$, then every prime factor of e is at least 5, so $e \ge 5^j \ge 4^j$, and (2.14) holds with $\kappa(N) = 1$. Otherwise $\kappa(N) = 2$. The smallest possible product of j distinct primes is at least 4^{j-1}. Hence $e \ge 4^{j-1}$, so $4^j d = 4^j N/e \le 4N$, proving (2.14).

Let

$$\mathcal{D}_j := \{d \mid N : d > 1,\ \omega(N/d) = j\}.$$

By (2.14),

$$\sum_{d\in\mathcal{D}_j} \sqrt{d}\log(4d)2^{\omega(N/d)} \le \kappa(N)\sqrt{N}\sum_{d\in\mathcal{D}_j}\log(4d). \tag{2.15}$$

The set $\mathcal{D}_j$ has $\binom{m}{j}$ elements. Moreover, each prime factor of N occurs in exactly $\binom{m-1}{j}$ of the divisors $d \in \mathcal{D}_j$, since it occurs in $d = N/e$ precisely when it is not chosen as one of the j prime factors of e. Therefore

$$\prod_{d\in\mathcal{D}_j} 4d = 4^{\binom{m}{j}} N^{\binom{m-1}{j}},$$

and hence

$$\sum_{d\in\mathcal{D}_j}\log(4d) = \binom{m-1}{j}\log N + \binom{m}{j}\log 4. \tag{2.16}$$

Combining (2.15) and (2.16), then summing over $0 \le j \le m-1$, gives

$$\begin{aligned}\sum_{\substack{d>1\\ d\mid N}} \sqrt{d}\log(4d)2^{\omega(N/d)} &\le \kappa(N)\sqrt{N}\left(\sum_{j=0}^{m-1}\binom{m-1}{j}\log N + \sum_{j=0}^{m-1}\binom{m}{j}\log 4\right)\\ &\le \kappa(N)\sqrt{N}\left(2^{m-1}\log N + 2^m\log 4\right)\\ &= \kappa(N)2^{m-1}\sqrt{N}\,(\log N + 4\log 2)\,,\end{aligned} \tag{2.17}$$

which is (2.13). □

We then obtain the following positivity criterion.

Corollary 2.18. *Let $m = \omega(N)$. If*

$$\frac{\varphi(N)}{12} > A(N) + \frac{\kappa(N)2^m}{\pi}\sqrt{N}\,(\log N + 4\log 2)\,, \tag{2.19}$$

then $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ for every sign pattern ε_N.

Proof. Combine Lemmas 2.5 and 2.12. □

Applying the positivity criterion. The upshot of (2.19) is that $A(N)$ and $\kappa(N)$ depend only on the number of prime divisors of N and the congruence class N modulo 6; we exploit this in the following statement.

Proposition 2.20. *The following statements hold.*

(i) *If* $N \equiv \pm 1 \pmod 6$, *then* $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ *if* $\omega(N) \geq 9$.
(ii) *If* $N \equiv 3 \pmod 6$, *then* $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ *if* $\omega(N) \geq 11$.
(iii) *If* $N \equiv 0 \pmod 2$, *then* $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ *if* $\omega(N) \geq 13$.

Proof. We prove that (2.19) holds in the indicated ranges.

We choose one of the three congruence classes. We let $m \geq 1$ and consider integers N in the given congruence class with $\omega(N) = m$. Let P_m be the product of the m smallest possible prime factors of N in that class: so for (i) we take the product of the first m primes $p \geq 5$; for (ii) we take 3 times the product of the first $m-1$ primes at least 5; and for (iii) we just take the product of the first m primes. Let

$$\rho_m := \prod_{p \mid P_m} \left(1 - \frac{1}{p}\right). \tag{2.21}$$

For any squarefree N in the congruence class with $\omega(N) = m$, we have $N \geq P_m$ by construction and

$$\varphi(N) = N \prod_{p \mid N} \left(1 - \frac{1}{p}\right) \geq \rho_m N. \tag{2.22}$$

For $x \geq 1$ define

$$F_m(x) := \frac{\rho_m}{12} x - A(m) - \frac{\kappa 2^m}{\pi} \sqrt{x} \left(\log x + 4 \log 2\right), \tag{2.23}$$

where $A(m)$ and κ are determined by m and the congruence class. By (2.22), the positivity criterion follows from $F_m(N) > 0$. We have

$$F_m'(x) = \frac{\rho_m}{12} - \frac{\kappa 2^m}{\pi} \frac{\log x + 4 \log 2 + 2}{2\sqrt{x}}. \tag{2.24}$$

The function $(\log x + 4 \log 2 + 2)/(2\sqrt{x})$ is decreasing for $x \geq 1$, thus to check $F_m(x) > 0$ for $x \geq x_0$ it suffices to show that $F_m(x_0) > 0$ and $F_m'(x_0) > 0$.

We first check the base values $m = 9, 11, 13$ for the three classes. At the minimal possible $x = P_m$, direct numerical evaluation gives

congruence	m	P_m	$F_m(P_m)$	$F_m'(N)$
$N \equiv \pm 1 \pmod 6$	9	33426748355	$> 4.7 \cdot 10^8$	> 0.025,
$N \equiv 3 \pmod 6$	11	3710369067405	$> 1.2 \cdot 10^{10}$	> 0.013,
$N \equiv 0 \pmod 2$	13	304250263527210	$> 3.9 \cdot 10^{11}$	> 0.006.

Thus $F_m(x) > 0$ for all $x \geq P_m$ in these base cases.

It remains to propagate the result to larger m which we do by induction on m. Suppose (2.19) holds for every N in one of the three classes with $\omega(N) = m$, where m is at least the corresponding base value. Let N^* be in the same class with $\omega(N^*) = m+1$, and let q be the

largest prime factor of N^*. Put $N = N^*/q$. Then N is still in the same class, and $q \geq 37$ in all three inductions. By the induction hypothesis,

$$\frac{\varphi(N)}{12} > A(m) + \frac{\kappa 2^m}{\pi}\sqrt{N}(\log N + 4\log 2).$$

Multiplying both sides by $q-1$, we get

$$\frac{\varphi(N^*)}{12} > (q-1)A(m) + \frac{\kappa 2^m (q-1)}{\pi}\sqrt{N}(\log N + 4\log 2).$$

Of course $(q-1)A(m) \geq 2A(m) > A(m+1)$; and we claim that for $q \geq 37$,

$$q - 1 \geq 2\sqrt{q}\left(1 + \frac{\log q}{4\log 2}\right) \tag{2.25}$$

since the difference is > 7 at $q = 37$ and the derivative is positive for $q \geq 37$. Since $\log N + 4\log 2 \geq 4\log 2$, (2.25) implies

$$(q-1)(\log N + 4\log 2) \geq 2\sqrt{q}(\log(qN) + 4\log 2).$$

Therefore

$$(q-1)\frac{\kappa 2^m}{\pi}\sqrt{N}(\log N + 4\log 2) \geq \frac{\kappa 2^{m+1}}{\pi}\sqrt{N^*}(\log N^* + 4\log 2). \tag{2.26}$$

Combining these two inequalities proves (2.19) for N^*. The proof is finished by induction. □

Computation. After Proposition 2.20, we are left with finitely many possible values of $\omega(N)$ in each congruence class.

Lemma 2.27. *For N in the given congruence class with given number of prime divisors $\omega(N)$, if $N > B$ in Table 1, then* $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$.

$\omega(N)$	$N \equiv \pm 1 \pmod 6$	$N \equiv 3 \pmod 6$	$N \equiv 0 \pmod 2$
13	-	-	-
12	-	-	52320255687514
11	-	-	11302808177135
10	-	584712825763	2402574532579
9	-	122979855890	505520992240
8	2544226303	25568947411	103627324753
7	518227848	5170085802	20662946692
6	102294401	1014636104	4029222669
5	19508619	194205148	745913629
4	3607458	35153388	131967432
3	625485	6050500	20186499
2	101953	892300	2588857

TABLE 1. Bounds B such that $N > B$ implies $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ for every sign pattern.

A dash in Table 1 means that Proposition 2.20 already gives positivity for all such N.

Proof. The same positivity criterion gives explicit cutoffs for N for fixed $m = \omega(N)$ and congruence class. We compute the associated upper bounds on N in Table 1, where an entry B means that every squarefree N in that congruence class with the indicated value of $\omega(N)$ and with $N > B$ has $\dim S_2^{\mathrm{new},\varepsilon_N}(N) > 0$ for every sign pattern ε_N. The entries in Table 1 are obtained by rounding up after solving $F_m(x) = 0$ where $F_m(x)$ is as in (2.23), for fixed m in a given congruence class. A numerical solution can be quickly estimated, then $F_m(B) > 0$ verified rigorously. Computing the derivative and repeating the argument in the proof of Proposition 2.20 then also shows that $F_m(N) > 0$ for all $N \geq B$. □

It remains to enumerate the finitely many levels below these cutoffs and then evaluate Martin's exact dimension formula. We recall from [Mar18, Propositions 1.4 and 3.2] that for $d > 1$

$$\dim S_2^{new,\epsilon_N}(N) = 2^{-\omega(N)} \sum_{d|N} \epsilon_N(d) \cdot \mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_d, \text{ where}$$

$$\mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_d := -\frac{1}{2} h'(\Delta_d) b(d, N/d) \prod_{\substack{p|(N/d) \\ p\neq 2}} \left(\left(\frac{\Delta_d}{p} \right) - 1 \right) + (-1)^{\omega(N)-\omega(d)}$$

$$- \frac{\delta[d=2]}{2} \cdot \prod_{p|(N/d)} \left(\left(\frac{-4}{p} \right) - 1 \right) - \frac{\delta[d=3]}{3} \cdot \prod_{p|(N/d)} \left(\left(\frac{-3}{p} \right) - 1 \right).$$

The computation proceeds with the following steps.

1. Fix a row and column of Table 1. Thus $m = \omega(N)$, the congruence class of N, and a bound B are fixed. If the table entry is a dash, there is nothing to enumerate.
2. Generate all increasing m-tuples of primes $(p_1, \ldots, p_m)$ in the required congruence class with $\prod_i p_i \leq B$. During the recursion, after $p_1, \ldots, p_i$ have been chosen, it is enough to consider
$$p_i < p_{i+1} \leq \left(\frac{B}{\prod_{k=1}^{i} p_k} \right)^{1/(m-i)}, \tag{2.28}$$
otherwise even choosing all remaining primes at least p_{i+1} would make the product exceed B.
3. Apply the filter from the starting inequality: if $\dim S_2^{\mathrm{new},\varepsilon_N}(N) = 0$ for some sign pattern, then by (2.7) we must have
$$\frac{\varphi(N)}{12} \leq A(N) + \frac{2}{\pi} \sum_{\substack{d>1 \\ d|N}} \sqrt{d} \log(4d) 2^{m-\omega(d)}. \tag{2.29}$$
We discard N if (2.29) fails.
4. Apply a sign-pattern filter. For a fixed sign pattern ε_N, Martin's trace formula identifies exactly which divisors d can contribute negatively. Let $S(\varepsilon_N)$ be the set of divisors $d \mid N$ satisfying one of the following conditions:
   a. N is odd, $d > 3$,
   $$\varepsilon_N(d) b(d, N/d) (-1)^{\omega(N/d)} = 1,$$
   and $(-d \,|\, p) = -1$ for all $p \mid N/d$;

b. N is even, $d \equiv 1, 3 \pmod 4$, $d > 3$,

$$\varepsilon_N(d) b(d, N/d) (-1)^{\omega(N/d)} = -1,$$

and $(-d \,|\, p) = -1$ for all primes $p \mid N/d$ with $p \neq 2$;

c. N is even, d is even, $d > 3$,

$$\varepsilon_N(d) b(d, N/d) (-1)^{\omega(N/d)} = -1,$$

and $(-d \,|\, p) = -1$ for all $p \mid N/d$.

Then a lacking pair satisfies the sharper necessary inequality

$$\frac{\varphi(N)}{12} \leq A(N) + \frac{2}{\pi} \sum_{d \in S(\varepsilon_N)} \sqrt{d} \log(4d) 2^{\omega(N/d)}. \tag{2.30}$$

This step discards sign patterns for which the negative trace terms are too small to cancel the main term.

5. For the remaining pairs (N, ε_N), compute $\dim S_2^{\mathrm{new},\varepsilon_N}(N)$ exactly using a Magma implementation [BCP97] of the Sage code [The22] supplied by Martin [Mar18].

Steps 2–4 are not logically necessary, but they really help—so much so that we can call the LMFDB [LMF20] for the class numbers we need by the time we get to the final step.

The resulting zero-dimensional eigenspaces are exactly those listed in Tables 2, 3, 4, and 5. The Magma enumeration and filtering in Steps 2–5 took 278234.59 seconds on a single 4 x Intel Xeon Gold 6148 CPU with 64 threads, with total memory usage 33901.19MB.

As an immediate corollary, we obtain the squarefree composite levels for which no weight 2 newform has root number -1.

Corollary 2.31. *Let $S_2^{\mathrm{new},-}(N)$ denote the subspace generated by newforms of weight 2 and squarefree level N with root number -1. Then there are exactly 45 squarefree composite integers N such that $\dim S_2^{\mathrm{new},-}(N) = 0$, namely*

$$6, 10, 14, 15, 21, 22, 26, 30, 33, 34, 35, 38, 39, 42, 46, 51, 55, 62, 66, 69, 70, 74, 78, 86, 87, 94, 95,$$
$$105, 110, 111, 114, 134, 146, 159, 174, 182, 186, 194, 195, 206, 222, 230, 231, 255, 266.$$

Proof. For a weight k newform of squarefree level N and Atkin–Lehner sign pattern ε_N, the root number is

$$(-1)^{k/2} \prod_{p \mid N} \varepsilon_N(p).$$

For $k = 2$, the root number is -1 exactly when $\prod_{p \mid N} \varepsilon_N(p) = 1$, equivalently $\varepsilon_N(p) = -1$ for an even number of prime divisors $p \mid N$. Therefore $S_2^{\mathrm{new},-}(N) = 0$ exactly when all sign eigenspaces in Tables 2, 3, 4, and 5 with this parity condition are zero. Reading off those levels gives the displayed list. □

Remark 2.32. A previous work of Erick Ross classifies levels N, weights k, and Dirichlet characters χ modulo N such that $\dim S_k^{\mathrm{new}}(\Gamma_0(N), \chi) = 0$ [Ros26, Theorem 1.2]. A result on proportion of forms in $S_k(N)$ or $S_k^{\mathrm{new}}(N)$ with the given Atkin–Lehner sign pattern is computed in [RVLWX26].

3. Gained primes of good reduction

As in the introduction, let D, N be coprime squarefree integers with D the product of an even number of primes (with $D = 1$ allowed). Let $X_0(D;N)$ be the modular or Shimura curve associated to an Eichler order of level N in a quaternion algebra of discriminant D, and let $X_0^*(D;N) := X_0(D;N)/W_{DN}$ be the full Atkin–Lehner quotient of the associated modular or Shimura curve. We say that a prime $p \mid DN$ is an *unexpected* or *gained* prime of good reduction if the curve $X_0^*(D;N)$ or its Jacobian $J_0^*(D;N) := \mathrm{Jac}(X_0^*(D;N))$ has good reduction at p.

In the case when the genus of $X_0^*(D;N)$ is $g = 0$, the Jacobian is trivial and automatically all primes $p \mid DN$ are gained primes of good reduction for the Jacobian. Whenever $X_0^*(D;N)_{\mathbb{Q}} \simeq \mathbb{P}^1_{\mathbb{Q}}$—for example whenever $D = 1$ and the cusp ∞ is $\mathbb{Q}$-rational, or more generally when there is a rational CM point—then again all primes are gained for the curve. This is always the case, as proven by [NR15, Proposition 4.2].

Semistable quotients and a one-vertex criterion. Let R be a discrete valuation ring with fraction field K and residue field k. Let X be a smooth, proper, geometrically connected curve over K with semistable reduction, and let $\mathcal{X}$ be a proper semistable model over R. We write $\Gamma_{\mathcal{X}}$ for the dual graph of the geometric special fibre $\mathcal{X}_{\bar{k}}$: its vertices are the irreducible components, its edges are the nodes, and a self-intersection on a component gives a loop.

Lemma 3.1. *Let $J := \mathrm{Jac}(X)$, and let $\mathcal{J}$ be its Néron model over R. Then the identity component of the geometric special fibre fits in an extension*

$$0 \to T \to \mathcal{J}^0_{\bar{k}} \to A \to 0,$$

where T is a torus and A is an abelian variety, and there is a canonical isomorphism of the character group of T with $H_1(\Gamma_{\mathcal{X}}, \mathbb{Z})$.

In particular, J has good reduction if and only if $\Gamma_{\mathcal{X}}$ is a tree, equivalently if and only if the special fibre is of compact type.

Proof. This is Raynaud's description of the relative Picard functor; see Raynaud [Ray70] and Bosch–Lütkebohmert–Raynaud [BLR90, Section 9.2, Theorem 8]. If the total space is not regular, resolving a node only subdivides the corresponding edge and therefore does not change H_1 of the dual graph. The final assertion is the usual compact-type criterion for good reduction of the Jacobian. □

Corollary 3.2. *Suppose that X has a semistable model whose geometric special fibre is irreducible. Then the following are equivalent:*

(i) *X has good reduction;*
(ii) *the dual graph has no loops; and*
(iii) *$\mathrm{Jac}(X)$ has good reduction.*

Proof. The dual graph has one vertex, so every edge is a loop and $b_1(\Gamma_{\mathcal{X}}) = \#E(\Gamma_{\mathcal{X}})$. Thus it is a tree if and only if it has no edges. Having no edges means that the irreducible special fibre has no nodes, hence is smooth. The result now follows from Lemma 3.1. □

We next verify that this criterion applies to the full Atkin–Lehner quotient.

Proposition 3.3. *Let $p \mid N$. Then the curve $X_0^*(D;N)$ has semistable reduction at p, and it has a semistable model over $\mathbb{Z}_p$ whose geometric special fibre is irreducible. In particular, $X_0^*(D;N)$ has good reduction at p if and only if $J_0^*(D;N)$ has good reduction at p.*

Proof. The integral model of $X_0(D;N)$ at p has two geometric irreducible components, each identified with the corresponding curve with the p-part of the level removed, and these components meet transversally at the supersingular points: for modular curves this is the Deligne–Rapoport model [DR73, Chapter V, Section 1]; see also Mazur [Maz77, Appendix]. For Shimura curves over $\mathbb{Q}$, the analogous statement is Buzzard [Buz97, Theorem 4.7]; see also Carayol [Car86]. In both cases the involution w_p interchanges the two components.

Let $\mathcal{X}$ be the semistable integral model of $X_0(D;N)$ over $\mathbb{Z}_p$ described above, and form the finite quotient $\pi\colon \mathcal{X} \to \mathcal{X}/W_{DN}$. The quotient $\mathcal{X}/W_{DN}$ is a normal proper flat curve over $\mathbb{Z}_p$, with generic fibre $X_0^*(D;N)$. By Liu–Lorenzini [LL99, Proposition 1.6], the curve $\mathcal{X}/W_{DN}$ is semistable.

As above, the special fibre of $\mathcal{X}$ has two irreducible components with w_p interchanging them. Since π is finite and surjective, the images cover the special fibre, so the special fiber is irreducible.

The final statement follows from Corollary 3.2. □

Automorphic criterion. By the Jacquet–Langlands correspondence, the space of regular differentials on $X_0(D;N)$ is

$$H^0(X_0(D;N),\Omega^1) \simeq S_2^{D\text{-new}}(\Gamma_0(DN)) \simeq \bigoplus_{D\mid M\mid DN} S_2^{\text{new}}(\Gamma_0(M))^{\oplus d(DN/M)} \tag{3.4}$$

decomposing into newspaces with multiplicity according to Atkin–Lehner theory. For M with $D \mid M \mid DN$, define the sign pattern ε_M by $\varepsilon_M(w_q) = -1, +1$ according as $q \mid D$ or $q \mid (M/D)$ (signs flip at ramified primes).

Taking W_{DN}-invariants removes the oldform multiplicities in (3.4). Indeed, for each prime $q \mid (DN/M)$ (necessarily $q \mid N$), the q-old degeneracy space is two-dimensional and w_q has one-dimensional eigenspaces for each sign. Selecting the invariant line at every such q leaves one copy of the prescribed eigenspace at level M; see Atkin–Li [AL78]. Consequently

$$H^0(X_0^*(D;N),\Omega^1) \simeq S_2^{D\text{-new},\varepsilon_{DN}}(\Gamma_0(DN)) \simeq \bigoplus_{D\mid M\mid DN} S_2^{\text{new},\varepsilon_M}(\Gamma_0(M)). \tag{3.5}$$

It follows then that the isotypic decomposition of $J_0^*(D;N)$ up to isogeny is

$$J_0^*(D;N) \sim_{\mathbb{Q}} \prod_{D\mid M\mid DN} \prod_{[f]} A_f, \tag{3.6}$$

where $[f]$ runs over Galois orbits of normalized newforms in $S_2^{\text{new},\varepsilon_M}(\Gamma_0(M))$ and A_f is the associated modular abelian variety.

We obtain the following criterion, extending Proposition 3.3.

Lemma 3.7. *$J_0^*(D;N)$ has good reduction at $p \mid DN$ if and only if*

$$\dim S_2^{\text{new},\varepsilon_M}(\Gamma_0(M)) = 0$$

for all M such that $D \mid M \mid DN$ and $p \mid M$.

In other words, unexpected good reduction for the Jacobian occurs exactly when all newspaces in the decomposition with level a multiple of p are lacking.

Proof. For $\ell \neq p$, the ℓ-adic Tate module of $J_0^*(D;N)$ is unramified at p if and only if the same is true for each A_f occurring in the isogeny decomposition. By the Néron–Ogg–Shafarevich criterion [ST68], this is equivalent to good reduction. For a modular abelian variety attached to a newform of squarefree level M, ramification at p occurs exactly when $p \mid M$. □

Corollary 3.8. *If $X_0^*(D;N)$ has genus > 0, then both $X_0^*(D;N)$ and $J_0^*(D;N)$ have bad reduction at all primes $p \mid D$.*

Proof. Since $J_0^*(D;N)$ has positive dimension, any form in level M will have $p \mid M$ so by the previous lemma, p is a prime of bad reduction. □

Proof of Theorem 1.2. Part (a) is proven in Proposition 3.3; part (b) is proven in Corollary 3.8. So we prove part (c). Lemma 3.7 gives a straightforward way to compute Tables 6 and 7 using Tables 2–5. Namely, for each squarefree level N' in the finite range produced by Theorem 1.1, and for all possible decompositions $N' = DN$ with D the product of an even number of primes, we get a sign pattern and determine the set of primes p which satisfy Lemma 3.7. Keeping exactly those pairs (D, N, p) gives Tables 6 and 7. □

Remark 3.9. We went further using our data, with the assistance of ChatGPT, and found all curves *intermediate* between $X_0(N)$ and the full Atkin–Lehner quotients $X_0^*(N)$ and $X_0^*(D;N)$ with unexpected primes of good reduction: see our GitHub repository [PPV26] for the complete list. Here are two examples from the list.

- A modular curve example is $X_0(74)/\langle w_{74}\rangle$, a genus 2 curve of conductor 37^2. In this case, the quotient $X_0^*(74)$ is a genus 1 curve of conductor 37.
- A Shimura curve example can be seen in [PS25, Example 5.1]. The curve
$$X_0(210;11)/\langle w_2, w_5, w_7, w_{33}\rangle$$
is a genus 1 curve with unexpected prime of good reduction at 11. The quotient $X_0^*(210;11)$ is a genus 0 curve.

4. Tables

We provide Tables 2, 3, 4, and 5 from Section 2, and Tables 6 and 7 from Section 3. We elaborate on the entries of Tables 6 and 7. The first column (respectively the first two columns) indicates the level of the modular curve. The next two columns indicate the genera of curves $X_0(N)$ and $X_0^*(N)$ (and respectively $X_0(D;N)$ and $X_0^*(D;N)$). We use abbreviations $g := g(X_0(N))$ (and respectively $g := g(X_0(D;N))$) and $g^* := g(X_0^*(N))$ (and respectively $g^* := g(X_0^*(D;N))$). The fourth (respectively the fifth) column indicates gained primes of good reduction. We can find such good primes by looking at the isotypic decomposition of the Jacobians of respective modular curves (abbreviated as J^*). By Proposition 3.3, such primes are also gained primes of good reduction for the modular curves (abbreviated as X^*). Some increments of primes of good reduction for modular Atkin–Lehner quotients $X_0^*(N)$ have been studied in previous literature, which we explain in the last column of Table 6. We omit the genus-zero quotients, since their Jacobians are trivial and therefore do not distinguish the reduction behaviour of the curve.

References

[AL78] A. O. L. Atkin and Wen Ch'ing Winnie Li. Twists of newforms and pseudo-eigenvalues of W-operators. *Invent. Math.*, 48(3):221–243, 1978.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990.

[Buz97] Kevin Buzzard. Integral models of certain Shimura curves. *Duke Math. J.*, 87(3):591–612, 1997.

[Car86] Henri Carayol. Sur la mauvaise réduction des courbes de Shimura. *Compositio Math.*, 59(2):151–230, 1986.

[DR73] Pierre Deligne and Michael Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II*, volume 349 of *Lecture Notes in Math.*, pages 143–316. Springer, Berlin, 1973.

[LL99] Qing Liu and Dino Lorenzini. Models of curves and finite covers. *Compositio Math.*, 118(1):61–102, 1999.

[LMF20] The LMFDB Collaboration. The L-functions and modular forms database. https://beta.lmfdb.org/NumberField/QuadraticImaginaryClassGroups, 2020. [Online; accessed 28 November 2024].

[Lou04] Stephane R. Louboutin. Explicit upper bounds for values at s=1 of Dirichlet L-series associated with primitive even characters. *Journal of Number Theory*, pages 118–131, 2004.

[Mar05] Greg Martin. Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$. *J. Number Theory*, 112(2):298–331, 2005.

[Mar18] Kimball Martin. Refined dimensions of cusp forms, and equidistribution and bias of signs. *J. Number Theory*, 188:1–17, 2018.

[Maz77] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, 47:33–186, 1977. With an appendix by Mazur and M. Rapoport.

[NR15] Joan Nualart Riera. *On the Hyperbolic Uniformization of Shimura Curves with an Atkin-Lehner Quotient of Genus 0*. PhD thesis, Universitat de Barcelona, 2015. Thesis (Ph.D.)–Universitat de Barcelona.

[PPV26] Oana Padurariu, Sun Woo Park, and John Voight. Github repository for "Unexpected primes of good reduction in quotients of modular and Shimura curves"'. https://github.com/swpark2008/Unexpected-Primes-of-Good-Reduction-ModularShimura, 2026.

[PS25] Oana Padurariu and Frederick Saia. Shimura curve Atkin–Lehner quotients of genus at most two. Arxiv preprint, 2025.

[Ray70] Michel Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, 38:27–76, 1970.

[Ros26] Erick Ross. Newspaces with nebentypus: an explicit dimension formula and classification of trivial newspaces. *J. Number Theory*, 278:317–352, 2026.

[RVLWX26] Erick Ross, Alexandre Van Lidth, Martha Rose Wolf, and Hui Xue. Proportion of atkin–lehner sign patterns and hecke eigenvalue equidistribution. *Mathematical Proceedings of the Cambridge Philosophical Society*, 181(1):869–884, 2026.

[ST68] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.

[The22] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2022. DOI 10.5281/zenodo.6259615.

[Wei09] Jared Weinstein. Hilbert modular forms with prescribed ramification. *Int. Math. Res. Not. IMRN*, 8:1388–1420, 2009.

Sign pattern ε_N	Values of N
$[1,1]$	35, 55, 95, 119
$[-1,-1]$	35, 55, 65, 85, 95, 115, 215, 77, 119, 161, 143
$[1,-1,-1]$	455
$[-1,-1,1]$	455

TABLE 2. Complete list of squarefree integers $N \equiv \pm 1 \pmod 6$ such that $\dim S_2^{\mathrm{new},\varepsilon_N} = 0$ with respective sign patterns.

Sign pattern ε_N	Values of N
$[1,1]$	15, 21, 33, 39, 51, 69, 87, 111, 159
$[1,-1]$	21, 57, 93
$[-1,1]$	15, 33
$[-1,-1]$	15, 21, 33, 39, 51, 57, 69, 87, 93, 111, 129, 159, 183, 237
$[1,1,1]$	105, 195, 255, 231
$[1,1,-1]$	165, 273
$[1,-1,1]$	105, 165
$[1,-1,-1]$	105, 165, 195, 255, 285, 435, 615, 231
$[-1,1,1]$	105
$[-1,1,-1]$	105, 165, 195, 255, 231, 273, 399
$[-1,-1,1]$	105, 165, 195, 255, 285, 435, 615, 231, 273, 399

TABLE 3. Squarefree integers $N \equiv 3 \pmod 6$ such that $\dim S_2^{\mathrm{new},\varepsilon_N} = 0$ with respective sign patterns.

Sign pattern ε_N	Values of N
$[1,1]$	6, 10, 14, 22, 26, 34, 38, 46, 62, 74, 86, 94, 134, 146, 194, 206
$[1,-1]$	6, 10, 22, 34, 58, 82
$[-1,1]$	6, 10, 14, 22, 46
$[-1,-1]$	6, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 106, 118, 122, 134, 146, 166, 178, 194, 202, 206, 314
$[1,1,1]$	30, 42, 66, 78, 114, 174, 186, 222, 366, 654, 70, 110, 130, 170, 230, 530, 182, 266
$[1,1,-1]$	30, 42, 66, 102, 138, 282, 498, 70, 130, 190, 310, 322, 418
$[1,-1,1]$	42, 78, 114, 258, 70, 130, 154
$[1,-1,-1]$	30, 42, 66, 78, 102, 114, 138, 174, 186, 222, 246, 282, 318, 354, 426, 534, 70, 110, 190, 230, 290, 310, 154, 182, 238, 266, 434, 518, 602, 286, 374, 494
$[-1,1,1]$	30, 78, 102, 190
$[-1,1,-1]$	30, 42, 66, 78, 102, 114, 138, 174, 186, 222, 246, 258, 402, 426, 474, 70, 110, 130, 170, 230, 290, 370, 154, 182, 238, 266, 434, 374, 494
$[-1,-1,1]$	30, 42, 66, 78, 102, 114, 138, 174, 186, 222, 246, 258, 282, 318, 354, 366, 402, 426, 438, 474, 498, 534, 582, 642, 654, 678, 822, 70, 110, 130, 170, 190, 230, 290, 310, 370, 410, 530, 610, 154, 182, 266, 322, 406, 518, 602, 286, 374, 418, 442
$[-1,-1,-1]$	30, 42, 78, 70

TABLE 4. Squarefree integers $N \equiv 0 \pmod 2$ such that $\dim S_2^{\mathrm{new},\varepsilon_N} = 0$ with respective sign patterns, $\omega(N) \leq 3$.

Sign pattern ε_N	Values of N
$[1,1,1,1]$	330, 510, 870, 1230, 546, 770
$[1,1,1,-1]$	210, 390, 1110, 1290, 858
$[1,1,-1,1]$	210, 330, 570, 462
$[1,1,-1,-1]$	210, 390, 510, 690, 1590, 546, 798, 770
$[1,-1,1,1]$	210, 330, 510, 690, 870, 1122
$[1,-1,1,-1]$	210, 330, 390, 510, 570, 930, 462, 714, 910
$[1,-1,-1,1]$	210, 330, 390, 510, 690, 1110, 462, 546, 714, 966, 1218, 858, 1326, 1190
$[1,-1,-1,-1]$	330
$[-1,1,1,1]$	210, 462, 798, 910
$[-1,1,1,-1]$	210, 330, 390, 570, 1770, 462, 546, 966, 770
$[-1,1,-1,1]$	210, 330, 390, 510, 570, 690, 930, 1410, 546, 714, 798, 1302, 1190
$[-1,1,-1,-1]$	462
$[-1,-1,1,1]$	210, 330, 390, 510, 570, 690, 870, 930, 1410, 462, 546, 714, 966, 1218, 858, 770
$[-1,-1,1,-1]$	330
$[-1,-1,-1,-1]$	210, 330, 390, 510, 570, 690, 870, 930, 1110, 1230, 1290, 1410, 1590, 1770, 2010, 2130, 462, 546, 714, 798, 966, 1218, 1302, 1722, 1974, 858, 1122, 1254, 1914, 1326, 1794, 1938, 770, 910, 1190
$[1,1,-1,1,-1]$	2310
$[1,-1,1,1,-1]$	2730
$[1,-1,-1,-1,-1]$	2310, 2730, 3990
$[-1,1,1,1,-1]$	3570
$[-1,1,1,-1,1]$	2310
$[-1,1,-1,-1,-1]$	2310, 4290
$[-1,-1,1,-1,-1]$	2310, 2730
$[-1,-1,-1,1,-1]$	2310, 2730, 3570, 3990
$[-1,-1,-1,-1,1]$	2310, 2730, 3570, 3990, 4290

TABLE 5. Squarefree integers $N \equiv 0 \pmod 2$ such that $\dim S_2^{\text{new},\varepsilon_N} = 0$ with respective sign patterns, $\omega(N) = 4, 5$.

N	g	g^*	p good for J^* and X^*	unexpected $\simeq$
74	8	1	2	$X_0(74)^* \simeq X_0(37)^*$
86	10	1	2	$X_0(86)^* \simeq X_0(43)^*$
111	11	1	3	$X_0(111)^* \simeq X_0(37)^*$
114	17	1	2	$X_0(114)^* \simeq X_0(57)^*$
130	17	1	2	N/A
134	16	2	2	$X_0(134)^* \simeq X_0(67)^*$
146	17	2	2	$X_0(146)^* \simeq X_0(73)^*$
159	17	1	3	$X_0(159)^* \simeq X_0(53)^*$
170	23	2	2	N/A
174	27	1	3	$X_0(174)^* \simeq X_0(58)^*$
182	25	1	2	$X_0(182)^* \simeq X_0(91)^*$
186	29	2	2	$X_0(186)^* \simeq X_0(93)^*$
194	23	3	2	$X_0(194)^* \simeq X_0(97)^*$
195	25	1	3	N/A
206	25	2	2	$X_0(206)^* \simeq X_0(103)^*$
222	35	1	$2, 3$	$X_0(222)^* \simeq X_0(37)^*$
230	33	2	2	$X_0(230)^* \simeq X_0(115)^*$
231	29	1	3	$X_0(231)^* \simeq X_0(77)^*$
255	33	2	3	N/A
266	37	2	2	$X_0(266)^* \simeq X_0(133)^*$
330	65	2	2	N/A
546	105	4	2	N/A

TABLE 6. Gained primes p of good reduction for modular Atkin–Lehner quotients $X_0^*(N)$.

N	D	g	g^*	p good for J^* and X^*
2	91	19	1	2
2	123	21	1	2
2	133	27	2	2
2	141	23	1	2
2	145	29	2	2
2	155	31	1	2
2	177	29	2	2
2	187	41	2	2
2	213	35	2	2
2	217	45	3	2
2	247	55	3	2
2	259	55	3	2
2	267	45	3	2
2	301	63	4	2
2	1155	121	1	2
2	1365	145	2	2
2	1995	217	3	2
3	142	23	1	3
3	145	37	2	3
3	158	27	1	3
3	205	53	3	3
3	1430	161	1	3
5	91	37	1	5
6	65	49	1	3
6	85	65	1	3
6	91	73	2	2
6	115	89	2	2
6	133	109	3	2
10	57	53	1	2
10	77	89	3	2
10	93	89	2	2
14	51	65	2	2
15	38	37	1	3
21	46	57	1	3
26	33	69	2	2
34	21	53	1	2
34	35	109	3	2
34	39	109	3	2
46	15	49	1	2
46	21	73	2	2
51	14	37	1	3
58	21	89	3	2
74	15	77	2	2
85	14	53	1	5
93	14	65	1	3
141	10	65	1	3

TABLE 7. Gained primes p of good reduction for Shimura curves $X_0^*(D;N)$.

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