# ERRATA AND ADDENDA: *ON COMPUTING BELYI MAPS*

JEROEN SIJSLING AND JOHN VOIGHT

This note gives some errata for the article *On explicit descent of marked curves and maps* [1].

## Errata

(1) In Remark 4.1.4, "marked maps curves" should be "marked maps".

## Addenda

The method of branches also gives a compact proof of the fact that a *Galois* Belyi map descends to its field of moduli, as a corollary of Theorem 3.2.4. (This descends the curves and the map, but not necessarily the action of the group.) More generally, we have the following.

**Corollary.** *Let $(Y, f\colon Y \to X; \mathcal{R})$ be a map of marked curves over $F^{\text{sep}}$ that is generically Galois. Suppose that $\mathcal{R}$ is nonempty, containing either a smooth point $Q \in Y(F^{\text{sep}})$ or a smooth point $P \in X(F^{\text{sep}})$. Then $(Y, f; \mathcal{R})$ descends to its field of moduli.*

*Proof.* Since $f\colon Y \to X$ is Galois, there exists a finite subgroup

$$H \leq \operatorname{Aut}(Y, f; \mathcal{R})(F^{\text{sep}})$$

and an isomorphism of marked maps from $(Y, f; \mathcal{R})$ to $(Y, \pi; \mathcal{R})$ where

$$\pi\colon Y \to H\backslash Y$$

is the canonical quotient map. We apply Theorem 3.2.4, with $H$ in place of $G$, and from $\mathcal{R}$ we take either take $S = P \in X(F^{\text{sep}})$ or $S = \pi(Q)$ if $Q \in Y(F^{\text{sep}})$. Repeating the proof of (a), the set $B(\pi, S)$ of branches is a $H$-torsor. Repeating the proof of (b), we obtain a Weil cocycle, so descent follows from Theorem 2.1.1.

In a bit more detail, let $b \in B(\pi, S)$ be a branch of $\pi$ whose underlying point $Q \in Y(F^{\text{sep}})$ is in the fiber above $S$. Given $\sigma$, there exists an automorphism $\phi_\sigma\colon \sigma(Y) \to Y$. Since the Galois group acts freely and transitively on the set of branches, we can ensure that $\phi_\sigma$ sends $\sigma(b)$ to $b$: since $\sigma(Q) \in \sigma(\pi)^{-1}(P)$, it is mapped to $Q$ by $\phi_\sigma$ by its defining property of transforming $(\sigma(X), \sigma(\pi))$ to $(Y, \pi)$:

$$\pi(\phi_\sigma(\sigma(Q))) = \sigma(\pi)(\sigma(Q)) = \sigma(\pi(Q)) = \sigma(\infty) = \infty.$$

(This property of staying in the fiber is crucial, and not formal!) The Galois property of the map $\pi$ can be used to rewrite the cocycle in a unique fashion, ensuring the compatibility

$$\phi_{\sigma\tau} = \phi_\sigma \sigma(\phi_\tau)$$

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as in the proof of (b), essentially because of the free transitive action again. To be completely explicit, the map on the left sends $\sigma(\tau(b))$ to $b$, whereas the map on the right sends $\sigma(\tau(b))$ to

$$\begin{aligned}(\phi_\sigma \sigma(\phi_\tau))(\sigma(\tau(b))) &= \phi_\sigma(\sigma(\phi_\tau)(\sigma(\tau(b)))) \\ &= \phi_\sigma(\sigma(\phi_\tau(\tau(b)))) = \phi_\sigma(\sigma(b)) = b\end{aligned}$$

as claimed. □

**Corollary.** *A Galois Belyi map* $(X, f\colon X \to \mathbb{P}^1; -; 0, 1, \infty)$ *over* $\overline{\mathbb{Q}}$ *descends to its field of moduli.*

*Proof.* This is a special case of the preceding corollary. □

## References

[1] Jeroen Sijsling and John Voight, *On explicit descent of marked curves and maps*, Res. Number Theory **2**:27 (2016), 35 pages.